**Dr. Z.'s Probability Lecture 21 Handout: Moment Generating Functions**

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**Important Definition** If $X$ is a random variable, and $t$ is a real variable (or an abstract symbol), the **Moment Generating Function** of $X$, denoted by $M_X(t)$ is defined by

$$M_X(t) \,=\, E[\,e^{tX}\,] \quad .$$

**Problem 21.1**: The value of a piece of factory equipment after five years of use is $1000\,(0.3)^X$, where $X$ is a random variable having moment generating function

$$M_X(t) = \frac{1}{1-3t} \quad , \quad for \quad t < \frac{1}{3} \quad .$$

Calculate the expected value of this piece of equipment after five years of use.

**Sol. of 21.1**: Recall the **important identity** (from precalc)

$$a = e^{\ln a} \quad ,$$

that enables you to express any $a^t$ in terms of the exponential function

$$a^X \,=\, e^{(\ln a)\,X} \quad .$$

We need $E[\,1000(0.3)^X\,]$, which is he same $1000E[\,e^{\ln(0.3)\,X}\,]$, so

$$E[1000(0.3)^X] \,=\, 1000E[(0.3)^X] \,=\, 1000E[\,e^{\ln(0.3)\,X}] = 1000M_X(\,\ln(0.3)\,) \quad .$$

Since $\ln(0.3) = -1.203972804 < \frac{1}{3}$, we can use the given expression for $M_X(t)$, and we get

$$M_X(\ln(0.3)) \,=\, \frac{1}{1-3(\ln(0.3))} = 0.2168295080 \quad ,$$

so finally

$$E[\,1000(0.3)^X\,] \,=\, 1000 \cdot M_X(\ln(0.3)) \,=\, \frac{1000}{1-3(\ln(0.3))} \,=\, 216.8295080\ldots \quad .$$

**Answer to 21.1**: The expected value of this piece of equipment after five years of use is $\frac{1000}{1-3(\ln(0.3))} = 216.8295080\ldots$

**Important Reminders from Calc2**:

• The **Maclaurin** series of a function $f(x)$, alias, the **Taylor series around** $x = 0$, is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad .$$

the sum of the series, when it makes sense (i.e. converges) is equals to $f(x)$ .

• The **Maclaurin** series of $f(x) = e^x$ is

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad .$$

• The **Maclaurin** series of $f(x) = (1+x)^a$, for any number $a$ is

$$\sum_{n=0}^{\infty} \binom{a}{n} x^n \quad ,$$

where $\binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!}$

Applying the Expectation operation to

$$e^{Xt} = \sum_{i=0}^{\infty} \frac{1}{n!}(Xt)^n = \sum_{i=0}^{\infty} \frac{1}{n!} X^n t^n \quad ,$$

We get the **important property**

$$M_X(t) = \sum_{i=0}^{\infty} \frac{1}{n!} E[X^n] t^n \quad .$$

Equivalently $E[X^n] = M_X^{(n)}(0)$, i.e. the $n$-th moment of $X$ is the $n$-th derivative of $M_X(t)$ evaluated at $t = 0$.

**Maple Tip**: To find the first $n$ terms of the Maclaurin expansion of a function $f$ of a variable $t$, in Maple, you do the command

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taylor(f,t=0,n+1) ;
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For example to get the first ten terms of $e^{t+t^2}$, you do

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taylor(exp(t+t**2),t=0,11) ;     .
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**Useful Notation:** if we denote $E[X^n]$ by $m_n$, then the important property is written:

$$M_X(t) = \sum_{i=0}^{\infty} m_n \frac{t^n}{n!} \quad .$$

This gives an **important recipe**, for finding the first few moments. In particular, the *first moment*, $E[X]$, (expectation, alias mean), and the second moment $E[X^2]$. This enables you to find the variance using the famous formula

$$Var(X) = E[X^2] \, - \, E[X]^2 = m_2 - m_1^2 \quad .$$

Here goes:

Do the Maclaurin expansion and look up the coefficients of $t$ (and multiply by 1), the coefficient of $t^2$ (and multiply by $2! = 2$), the coefficient of $t^3$ (and multiply by $3! = 6$),etc.

**Problem 21.2**: Find the first three moments, and the variance, of a random variable $X$ whose moment generating function is given by

$$M_X(t) = \frac{1}{\sqrt{1-2t)}} \quad .$$

**Sol. of 21.2**: Writing $M_X(t)$ as $(1-2t)^{-1/2}$, and using the Binomial theorem, we have

$$(1-2t)^{-1/2} \; = \; 1 + \binom{-1/2}{1}(-2t) + \binom{-1/2}{2}(-2t)^2 + \binom{-1/2}{3}(-2t)^3 + \dots$$

$1+(-1/2)(-2t)+((-1/2)(-3/2)/2)(-2t)^2+((-1/2)(-3/2)(-5/2)/6)(-2t)^3+\ldots = 1+t+(3/2)t^2+(5/2)t^3+\ldots$

Hence

$$m_1 = 1 \quad , \quad \frac{m_2}{2} = \frac{3}{2} \quad , \quad \frac{m_3}{6} = \frac{5}{2} \quad ,$$

so

$$m_1 = 1 \quad , \quad m_2 = 3 \quad , \quad m_3 = 15 \quad .$$

Also $Var(X) = m_2 - m_1^2 = 3 - 1^2 = 3 - 1 = 2$.

**Ans. to 21.2**: $E[X] = 1$, $E[X^2] = 3$, $E[X^3] = 15$ and $Var(X) = 2$.

**Problem 21.3**: Find the first four moments, and the variance, of a random variable $X$ whose moment generating function is given by

$$M_X(t) = e^{t^2/2 + t^3/6} \quad .$$

**Sol. of 21.3**: (Corrected Dec. 7, 2017, thanks to Vincent Wang)

Writing $e^{t^2/2+t^3/6} \; = \; e^{t^2/2} \cdot e^{t^3/6}$, and expanding up to the power $t^4$, (whenever we encounter a power larger than 4 we ignore it, i.e. replace it by $\ldots$, since we only care about the first four moments.) we get

$$M_X(t) = e^{t^2/2} \cdot e^{t^3/6} = \left(1 + t^2/2 + \frac{1}{2}(t^2/2)^2 + \ldots\right) \cdot \left(1 + t^3/6 + \ldots\right)$$

$$1 + \frac{1}{2}t^2 + \frac{1}{8}t^4 + \ldots + \frac{1}{6}t^3(1 + \frac{1}{2}t^2 + \ldots)$$
$$= 1 + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{8}t^4 + \ldots$$

Hence
$$m_1 = 0 \quad , \quad \frac{m_2}{2} = \frac{1}{2} \quad , \quad \frac{m_3}{6} = \frac{1}{6} \quad , \quad \frac{m_4}{24} = \frac{1}{8} \quad .$$

So
$$m_1 = 0 \quad , \quad , m_2 = 1 \quad , \quad m_3 = 1 \quad , \quad m_4 = 3 \quad .$$

Also $Var(X) = m_2 - m_1^2 = 1 - 0^2 = 1$.

**Ans. to 21.3**: $E[X] = 0$, $E[X^2] = 1$, $E[X^3] = 1$, $E[X^4] = 3$, and $Var(X) = 1$.

**Problem 21.4**: An actuary determines that the claim size for a certain class of accidents is a random variable $X$, with moment generating function
$$M_X(t) = \frac{1}{(1-100t)^3} \quad .$$

Determine the standard deviation of the claim size for this class of accidents.

**Sol. to 21.4**: By the binomial theorem applied to $(1-100t)^{-3}$ we get
$$M_X(t) = 1 \,+\, (-3)\cdot(-100t)^1 \,+\, ((-3)(-4)/2)\cdot(-100t)^2 + \ldots$$
$$= 1 \,+\, 300\,t \,+\, 60000\,t^2 \quad .$$

Hence
$$m_1 = 300 \quad , \quad \frac{m_2}{2} = 60000 \quad .$$

Hence
$$m_1 = 300 \quad , \quad m_2 \,=\, 120000 \quad .$$

Hence
$$Var(X) = m_2 - m_1^2 \,=\, 120000 - 300^2 = 120000 - 90000 = 30000 \quad .$$

**Finally**, the standard deviation is $\sqrt{Var(X)} = \sqrt{30000} = \sqrt{3}\cdot 100 = 173.2050808\ldots$ .

**Ans. to 21.4**: The standard deviation of the claim size for this class of accidents is $\sqrt{3}\cdot 100 = 173.2050808\ldots$.

**Warning**: Do not confuse variance and standard deviation! Read the question carefully.

**Moment Generating Functions of Famous Distributions**

• If $X$ is a binomial random variable with parameters $n$ and $p$, then
$$M_X(t) \,=\, (pe^t + 1 - p)^n \quad .$$

• If $X$ is a Poisson random variable with parameter $\lambda$,

$$M_X(t) \,=\, e^{\lambda\,(e^t-1)} \quad .$$

• If $Z$ is a the standard normal random variable then

$$M_Z(t) \,=\, e^{t^2/2} \quad .$$

In particular, $m_2 = 1, m_4 = 3, m_6 = 15$, and, in general $m_{2r} = (2r)!/(2^r r!)$. Of course, the expectation, and **all** the odd moments are all always zero.

• If $X$ is a a normal random variable with parameters $\mu$ and $\sigma^2$,

$$M_X(t) \,=\, e^{\frac{\sigma^2\,t^2}{2}+\mu t} \quad .$$

**Important property of Moment Generating Functions**

If $X$ and $Y$ are **independent** random variables, then

$$M_{X+Y}(t) = M_X(t)\,M_Y(t) \quad .$$

More generally, if $X_1, X_2, \ldots X_k$ are **mutually independent** then

$$M_{X_1+\ldots+X_k}(t) = M_{X_1}(t)\cdots M_{X_k}(t) \quad .$$

**Note**: This implies the following facts that the we already know (but now we have a slick proof).

• The sum of two independent binomial random variables of parameters $(n, p)$ and $(m, p)$ is a a binomial random variable with parameter $(m + n, p)$

**Proof**:

$$(pe^t + 1 - p)^n \cdot (pe^t + 1 - p)^m \,=\, (pe^t + 1 - p)^{n+m} \quad .$$

(This is also obvious from common sense: If you toss a coin whose prob. of Heads is $p$, $m$ times, and then $n$ times, at the end it is like tossing it $m + n$ times).

• The sum of two **independent** Poisson random variables with parameters $\lambda_1$ and $\lambda_2$ is a Poisson random variable with parameter $\lambda_1 + \lambda_2$

**Proof**:

$$e^{\lambda_1\,(e^t-1)} \cdot e^{\lambda_2\,(e^t-1)} \,=\, e^{(\lambda_1+\lambda_2)\,(e^t-1)}$$

• The sum of two **independent** Normal random variables with parameters $(\mu_1, \sigma_1^2)$ and $(\mu_2, \sigma_2^2)$ is a Normal random variable with parameters $(\mu_1 + \mu_2\,,\, \sigma_1^2 + \sigma_2^2)$.

**Proof**:

$$e^{\frac{\sigma_1^2\,t^2}{2}+\mu_1 t}\cdot e^{\frac{\sigma_2^2\,t^2}{2}+\mu_2 t} \;=\; e^{\frac{(\sigma_1^2+\sigma_2^2)\,t^2}{2}+(\mu_1+\mu_2)t} \quad .$$

**Problem 21.5**: A company insures homes in three cities, New York, Chicago and LA. Since sufficient distance separates the cities, it is reasonable to assume that the losses occurring in these cities are independent.

The moment generating functions for the loss distributions of the cities are

$$M_{NY}(t) = (1-3t)^{-1.5} \quad ,$$

$$M_{CHICAGO}(t) = (1-3t)^{-2.5} \quad ,$$

$$M_{LA}(t) = \;(1-3t)^{-3} \quad .$$

Let $X$ represent the combined losses from the three cities. Calculate the standard deviation of $X$ and $E[X^3]$.

**Sol. to 21.5**: By **independence**,

$$M_X(t) = (1-3t)^{-1.5}\cdot(1-3t)^{-2.5}\cdot(1-3t)^{-3} \;=\; (1-3t)^{-7} \quad .$$

By the Binomial theorem

$$(1-3t)^{-7} = 1 + (-7)(-3t) + ((-7)(-8)/2)(-3t)^2 + ((-7)(-8)(-9)/6)(-3t)^3 + \ldots .$$

$$= 1 \;+\; 21\,t \;+\; 252\,t^2 + 2268\,t^3 + \ldots \quad .$$

Hence

$$m_1 = 21 \quad , \quad \frac{m_2}{2} = 252 \quad , \quad \frac{m_3}{6} = 2268 \quad .$$

Hence

$$m_1 = 21 \quad , \quad m_2 = 504 \quad , \quad m_3 = 13608 \quad .$$

It follows that the variance, $Var(X)$, is $m_2 - m_1^2 = 504 - 21^2 = 63$ and hence the standard deviation is $\sqrt{63}$.

**Ans. to 21.5**: The standard deviation of $X$ is $\sqrt{63}$ and $E[X^3]$ is 13608.

**Important Definition**: A continuous random variable with mean $\mu$ is **symmetric** if its density function $f(t)$ satisfies $f(\mu+t) = f(\mu-t)$. In particular if the mean is 0, $f(t)$ must be an **even function**, i.e. it satisfies

$$f(-t) = f(t) \quad ,$$

i.e., replacing $t$ by $-t$ (and simplifying) gives you the same thing.

**Obvious but Important Fact**: If $X$ is a continuous **symmetric** random variable with mean 0, 0, then **all** its odd moments are zero. In other words, the moment generating function $M_X(t)$ is also **even** (i.e. only contains even powers).

**Not so obvious but Important Fact**: If $X$ is a continuous random variable whose moment generating function, $M_X(t)$ is an even function of $t$ (in other words $M_X(-t) = M_X(t)$), then it is a symmetric random variable with mean 0. In particular $P\{X \geq 0\} = P\{X \leq 0\} = \frac{1}{2}$.

**Problem 21.6**: Let $X$ and $Y$ be identically distributed independent random variables such that the moment generating function of $X + Y$ is

$$M(t) = 0.1\, e^{-3t} \,+\, 0.8 \,+\, 0.1\, e^{3t} \quad , \quad for \quad -\infty < t < \infty \quad .$$

Calculate $P\{X \geq 0\}$.

**Sol. to 21.6**: Superficially, it looks like insufficient information. In general, there is no direct quick way (at least for you!) to deduce directly probabilities like $P\{X \geq a\}$ for arbitrary $a$. But "by symmetry", for $P\{X \geq 0\}$, a reasonable guess would be $\frac{1}{2}$.

Let's justify this guess. We have (by independence) $M_{X+Y}(t) = M_X(t)\, M_Y(t)$ and since they are **identically distributed** $M_X(t) = M_Y(t)$,

$$M_X(t)^2 = 0.1\, e^{-3t} \,+\, 0.8 \,+\, 0.1\, e^{3t} \quad , \quad for \quad -\infty < t < \infty \quad ,$$

that implies

$$M_X(t) \;=\; \sqrt{0.1\, e^{-3t} \,+\, 0.8 \,+\, 0.1\, e^{3t}} \quad .$$

Since $M_X(t) = M_X(-t)$, $M_X(t)$ is an even function, and hence $X$ is symmetric random variable with mean 0, and so $P\{\, X \leq 0\,\} \;=\; P\{\, X \geq 0\,\}$. Since they add-up to 1 we get that $P[X \leq 0] = \frac{1}{2}$.

**Ans. to 21.6**: $P\{X \geq 0\} = \frac{1}{2}$.

**Important Concept: Joint Moment Generating Function**: If $X_1$ and $X_2$ are two random variables then the **joint moment generating function** $M(t_1, t_2)$ (that depends on two variables) is defined by

$$M(t_1, t_2) \;=\; E[\, e^{t_1 X_1 + t_2 X_2}] \quad .$$

**Warning**: In general it is **false** that $M(t_1, t_2) = M_{X_1}(t_1)\, M_{X_2}(t_2)$. This happens only to be true if $X_1$ and $X_2$ are **independent**. In fact, that's how you can tell whether or not they are independent.

**Note**: Obviously $M(t_1, 0) = M_{X_1}(t_1)$ and $M(0, t_2) = M_{X_2}(t_2)$.

**Note**: If you have $k$ random variables, $X_1, \ldots, X_k$; then the joint moment generating function is the $k$-variable function

$$M(t_1, \ldots, t_k) \;=\; E[\, e^{t_1 X_1 \,+\, \ldots \,+ t_k X_k}] \quad .$$

**Problem 21.7:** $X$ and $Y$ are independent random variables with common moment generating function $M(t) = e^{t^2/2}$. Let $W = X + 2Y$ and $Z = Y - 3X$. Determine the joint moment generating function, $M(t_1, t_2)$ of $W$ and $Z$.

**Sol. to 21.7**: By **definition**:

$$M(t_1, t_2) \;=\; E[\, e^{t_1 W + t_2 Z}\,] \quad .$$

Let's simplify $t_1 W + t_2 Z$:

$$t_1 W + t_2 Z = t_1(X + 2Y) + t_2(Y - 3X) = (t_1 - 3t_2)\, X \;+\; (2t_1 + t_2)\, Y \quad .$$

Hence

$$M(t_1, t_2) \;=\; E[\, e^{(t_1 - 3t_2)\, X \,+\, (2t_1 + t_2)\, Y}\,] \quad .$$

Since $X$ and $Y$ are **independent**, we have

$$M(t_1, t_2) \;=\; E[\, e^{(t_1 - 3t_2)\, X}]\, E[e^{(2t_1 + t_2)\, Y}\,] \quad .$$

$$= M_X(t_1 - 3t_2)\, M_Y(2t_1 + t_2) \quad .$$

Recall that $M_X(t) = M_Y(t) = e^{t^2/2}$, hence

$$M(t_1, t_2) \;=\; e^{(t_1 - 3t_2)^2/2}\, e^{(2t_1 + t_2)^2/2} \;=\; e^{((t_1 - 3t_2)^2 + (2t_1 + t_2)^2)/2}$$

$$=\; e^{(t_1^2 - 6t_1 t_2 + 9t_2^2 + 4t_1^2 + 4t_1 t_2 + t_2^2)/2} \;=\; e^{(5t_1^2 - 2t_1 t_2 + 10t_2^2)/2} \;=\; e^{\frac{5}{2}t_1^2 - t_1 t_2 + 5t_2^2} \quad .$$

**Ans. to 21.7**: The joint moment generating function of $W$ and $Z$ is $e^{\frac{5}{2}t_1^2 - t_1 t_2 + 5t_2^2}$ .