Solutions to Real Quiz # 8 for Dr. Z.'s MathHistory

1. (2 points) What is Laplace's partial differential equation? Who derived it before Laplace?

Ans.
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$
Euler.

2. (2 points) What is the name of the city where Carl Friedrich Gauss was born? What was the occupation of his father?

Ans. Brunswick (like our city, New Brunswick!) (in German: Braunschweig). Day Laborer.

3. (1 point) Can the sides of a regular polygon of 17 sides be constructed with compass and rule alone? Who proved (or disproved) it?

Ans. Yes. Gauss **proved** it.

4. (5 point) In a perfect, platonic, solid, every vertex has the same number of edges adjacent to it, and every face has the same number of edges surrounding it (but of course, these two numbers do not have to be the same [although they may]).

Let, as usual,

- V be the number of vertices
- E be the number of edges
- F be the number of faces

If you call the number of edges meeting every vertex a, and the number of edges around every face b,

(i) Prove that $V = \frac{2E}{a}, F = \frac{2E}{b}$.

(ii) By using $V - E + F = 2$, and algebra, express E in terms of a and b.

(iii) Find all the $a, b \geq 3$ that makes E a positive integer. For each such choice, find V, E, and F, and name the corresponding perfect polyhedron.

Sol.

(i)

Every vertex has a edges coming out of it. Since there are v edges, altogether,by adding them up, we get a count of aV edges. But every edges belongs to exactly two vertices, so in the above, every edge got counted twice. So the actual number of edges is half of that, i.e. $E = \frac{aV}{2}$, hence $V = \frac{2E}{a}$.

Every face has b edges around it. Since there are F faces, altogether,by adding them up, we get a count of bF edges. But every edges belongs to exactly two faces, so in the above, every edge got counted twice. So the actual number of edges is half of that, i.e. $E = \frac{bF}{2}$, hence $F = \frac{2E}{b}$.

(ii) Since, from (i) we have $V = \frac{2E}{a}, F = \frac{2E}{b}$, using $V - E + F = 2$, we get

$$\frac{2E}{a} - E + \frac{2E}{b} = 2$$

Factoring the left

$$E\left(\frac{2}{a} - 1 + \frac{2}{b}\right) = 2$$

Hence

$$E = \frac{2}{\frac{2}{a} - 1 + \frac{2}{b}}.$$

a and b must be at least 3 (to make geometrical sense, the smallest polygon, the triangle, has three edges), but they can't be too big. By plugging-in $a = 3, 4, 5, \ldots$ and $b = 3, 4, 5$ we see that very soon e is infinity ($a = 4, b = 4$) and after that, worse, negative! Since e must be a positive **integer**, by plugging-in, only the following values of a and b are possible.

- $a = 3, b = 3$: then, $E = 6$, $V = 4$, $F = 4$, **Tetrahedron**.
- $a = 3, b = 4$: then $E = 12$, $V = 8$, $F = 6$, **Hexahedron** (alias cube).
- $a = 3, b = 5$: then, $E = 30$, $V = 20$, $F = 12$, **Dodecahedron** .
- $a = 4, b = 3$: then $E = 12$, $V = 6$, $F = 8$, **Octahedron** .
- $a = 5, b = 3$: then, $E = 30$, $V = 12$, $F = 20$, **Icosahedron**.