Solutions to Real Quiz # 3 for Dr. Z.'s MathHistory

1. (1 points) What was the most important discovery ascribed to the Pythagoreans?

Ans. to 1: The irrationality phenomenon, most famously that $\sqrt{2}$ is irrational.

2. (2 points) Describe the Dichotomy Paradox.

Ans. to 2: In order to get from your starting point to your destination, you must first visit the half-way point. Having gotten to that half-way point, there is a new one, and now you are $\frac{1}{2} + \frac{1}{4}$ away from the start. And yet another one, and now you are $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ from the start, and so on and so forth. No matter how many previous half-points that you have crossed, there is always another one, hence you will **never** make it to your destination.

3. (2 point) What method was the Platonic School's answer to Zeno's paradoxes? What Axiom did they use?

Ans. to 3: The method of Exhaustion, Archimedes' axiom.

4. (5 points) Jill runs 9 mph and Jack walks 3 mph. They start a race, but since Jill is so much faster, she lets Jack start 18 miles head.

(i) Spell out the first four steps of Zeno's proof that Jill will always be behind Jack, and *never* catch-up, and end it with the phrase *and so on and so forth*.

(ii) Prove Zeno wrong by using high-school algebra to figure out the exact time when they meet, and the distance of that meeting place from the starting point.

(iii) By summing an *infinite* (but convergent) power series, use Zeno's analysis to get the same answer (for the time) in (ii). Explain the Modern Calculus 'resolution' of Zeno's paradox.

Sol. to 4:

4(i) At time $t = 0$,

• Jill's Distance from Starting Point is 0 miles ;

• Jack's Distance from Starting Point is 18 miles .

Jill arrives at Jack's initial location, after $18/9 = 2$ hours, but by then Jack has advanced $3 \cdot 2 = 6$ extra miles. So

At time $t = 2$,

• Jill's Distance from Starting Point is 18 miles ;

• Jack's Distance from Starting Point is $18 + 6 = 24$ miles .

To get to the 24 miles point, Jill needs $\frac{6}{9} = \frac{2}{3}$ extra hours, but by then Jack has advanced $\frac{2}{3} \cdot 3 = 2$ extra miles. So

At time $t = 2 + \frac{2}{3}$,

• Jill's Distance from Starting Point is $18 + 6 = 24$ miles ;

• Jack's Distance from Starting Point is $18 + 6 + 2 = 26$ miles .

To get to the 26 miles point, Jill needs $\frac{2}{9}$ extra hours, but by then Jack has advanced $\frac{2}{9} \cdot 3 = \frac{2}{3}$ extra miles. So

At time $t = 2 + \frac{2}{3} + \frac{2}{9}$,

• Jill's Distance from Starting Point is $18 + 6 + 2 = 26$ miles ;

• Jack's Distance from Starting Point is $18 + 6 + 2 + \frac{2}{3} = 26\frac{2}{3}$ miles .

Continuing, whenever Jill caught up to the previous location of Jack, Jack would have advanced a little further, hence (according to Zeno), **Jill will never catch up**.

4(ii) Jill's distance from the start, at time t hours, is $0 + 9t$. Jack's distance from the start, at time t hours, is $18 + 3t$.

Let T be the time they meet. Then, by definition of *meet*,

$$9T = 18 + 3T \quad .$$

Solving for T:

$$6T = 18 \quad ,$$

giving $T = 3$.

Answer to 4(ii): In spite of Zeno, they meet-up after three hours.

4(iii): Assuming that the steps in Zeno's paradox can be done *infinitely many times*, the meeting time is the 'infinite sum'

$$2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \ldots = 2 \cdot \left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots\right) = 2 \sum_{i=0}^{\infty} (\frac{1}{3})^i \quad .$$

By modern Calculus, one 'makes sense' of this infinite series, by saying that its partial sums 'converge to a limit', and that limit is **declared** to be the **value** of the infinite sum.

By the famous formula

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \quad ,$$

(valid for x between -1 and 1), with $x = \frac{1}{3}$ we get

$$\sum_{i=0}^{\infty} (\frac{1}{3})^i = \frac{1}{1 - \frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2} \quad .$$

Hence the meet-up time is $2 \cdot \frac{3}{2} = 3$ hours, in agreement with 4(ii).