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MATH 436

Final Project

**History of Complex Numbers**

The field of mathematics is one full of logic and reasoning along with problem solving. However, how is it possible that something imaginary could be part of mathematics and be applicable to the world we live in? The truth is that without complex numbers the world would not be what it is today. This is the story of how complex numbers came to be and how they caused mathematics to grow which led to extraordinary results.

The journey of complex numbers started in 1545 by the mathematician Gerolamo Cardano. He had come up with a method for solving general cubic equations of the form . However, he would use certain combinations of numbers in order to avoid negative numbers. In particular, he used the following forms; , and so that he wouldn’t come across negative numbers. Cardano didn’t believe in negative numbers. The reason why he did this was because when he was solving these equations he sometimes got square roots of negative numbers which he said had to be imaginary. He used the notation even though he couldn’t find a way to formulate this completely and thought that there was no use for it anyway. Cardano had written his results in the *Ars Magna* in 1545.

Next, came the mathematician Rafael Bombelli who helped with formulating complex numbers while doing algebra in 1572. He had no issues using negative numbers and their multiplication properties. He is the first to opening the doors to complex numbers by which he created some rules for them. Also, he did examples of adding and multiplying complex numbers. He was ahead of his time by being the only mathematician to believe in complex numbers while other mathematicians who also lived during this period did not believe in them, including Cardano.

Then came John Wallis who was the first person to come up with a geometrical representation of complex numbers. Although, oddly he believed that negative numbers were greater than infinity but were not less than zero. But, perhaps because of this strange belief it led to furthering the theory of complex numbers. It is uncertain whether or not this is true.

Leonhard Euler though had stated that a number is either greater than zero, less than zero, or equal to zero. From this logic, he concluded that square roots of negative numbers do not belong in any of these three categories and therefore must be imaginary since they aren’t real numbers since they only exist in the imagination of the mind.

Caspar Wessel gave the modern geometric interpretation of complex numbers. But, his work didn’t get recognized until a French translation appeared in 1897. Wessel also treated complex numbers as vectors even though he didn’t use the term vector and derived most of the properties such as multiplication in trigonometric form.

In 1799, Carl Friedrich Gauss in his Ph. D thesis gave the first proof of the *Fundamental Theorem of Algebra* which states that *Every polynomial equation of degree n with complex coefficients has n roots in the complex numbers.* However, Euler, Lagrange, and D’Alembert used complex numbers to try to prove it but were not successful.Although, in 1825 Gauss said that “the true metaphysics of is illusive.” In 1831 Gauss had vanquished his doubts on this by an application of complex numbers to *Number Theory*. This application to *Number Theory* was a major breakthrough to the acceptance of complex numbers. Although, there still wasn’t a complete understanding of complex numbers throughout the world.

For Instance, Rene Descartes had said that negative roots were false even though ironically, he coined the term imaginary. Also, Augustus De Morgan who was a famous logician and mathematician said that the  and -b being solutions to a problem indicates that there is an error somewhere. He also stated that 0-a and are both equally imaginary and inconceivable. However, there were still mathematicians who believed in complex numbers. Johann Lambert had used complex numbers for map projection and D’Alembert had used complex numbers in hydrodynamics.

In 1707, Abraham De Moivre claimed that he had discovered what he called his “magic” formula by doing numerical examples. In 1730, he had a general formula in the form of . Then in 1738 he had explained a difficult way to find roots in the form of which led to his famous formula *De Moivre’s Theorem* which is defined as . This formula led to further work in the complex numbers, more specifically the *Theory of Complex Functions*.

In 1814, Augustin-Louis Cauchy had obtained the famous *Cauchy-Riemann equations*, and by working with the interchange of double integrals in the real number plane. These equations are important because when they are satisfied for then is an analytic function meaning that it is differentiable at every point in a region. Although, Cauchy had pointed out that his equations have the entire theory of going from the real plane to the complex plane. However, he did not make his equations the foundation of his function theory. Although, Bernhard Riemann did just the opposite by using the *Cauchy-Riemann equations* at the beginning of his function theory and always built off them. Interestingly enough neither Cauchy or Riemann were the first to discover these equations. In fact, they appeared in prior years such that in 1752 in D’Alembert’s theory of fluids in his work *Essai d’une nouvelle theorie de la resistance des fluids*. They also appeared in the works of Lagrange and Euler. This is an intricate part of the *Theory of Complex Functions*.

In 1825 Gauss was the first to produce written work on Conformal mappings, which are maps that preserve angles and they have domains and ranges in the complex plane. Gauss had figured out that angle preserving mappings between the domains of the plane where is equivalent to the complex plane could be explained by holomorphic or non-holomorphic functions. Riemann on the other hand used conformal mappings in a different way. What he did was that he took two functions and as points in two different planes A and B respectively. From here he made a relation between the two planes. More specifically, there was similarity between the tiniest parts of plane A and the images of plane B. However, Cauchy and Weierstrass did not used conformal mappings in their work.

The next important development in *Complex Analysis* is that of infinite series. Infinite series is not deals with functions in the real plane but also in the complex plane. One of the most important series in the complex plane is power series. Euler’s famous formula for all could be expressed as a power series by taking the power series for, , , and and substituting each of these series into Euler’s formula respectively. Also, it gives us another way to plot points in the complex plane. A more important role of Euler’s formula is that of representing and in terms of such that and . These two formulas give another representation of these two trigonometric functions. Another, interesting outcome is that the hyperbolic functions and in the real plane also exist in the complex plane that of similar to and in terms of . Thus, hyperbolic cosine and sine are represented as and . Similarly, each of these two functions also have power series and give us the sum formulas and . So, because of these two formulas the complex cosine and sine functions can be broken down into their real and imaginary parts.

A crucial part to *Complex Analysis* was the domain of logarithmic functions in the complex plane. There is the permanence principle which states that functional relations that hold in the real plane should also hold in the complex plane. In the early eighteenth century people believed that there existed a function that satisfied two equations and . However, in 1702 Johann Bernoulli had known about an extraordinary equation . This was a huge discovery because it showed that the product of and was a real number. It’s hard to believe that something like this was actually true. Although, in 1749 Euler had a question of his own about the permanence principle pertaining to logarithms. He said that every number has infinitely many logarithms. From here he said “We see therefore that it is essential to the nature of logarithms that each number have an infinity of logarithms and that all these logarithms be different, not only from one another, but also[different] from all other logarithms of every other number.” (Remmert 159) This discovery made by Euler helped further the understanding that functions in the real plane and the complex plane do not always have the same domain.

After analyzing and discovering differentiability in the complex plane led to integration in the complex plane. It started with S.D. Poisson who was the first to integrate in the complex plane and published his work in 1813. Cauchy though was the first to make systematic investigations of integration in the complex plane. From his investigations, he thought for a very long time about how to integrate complex functions. At first, he would integrate the real and imaginary parts of the function separately. But, he realized that it was easier to integrate complex functions without separation and to combine the two integrals and which gives with , where f(z) must be analytic in some simply connected region. Also, must be contained within the same region. This result became known as the *Cauchy Integral Theorem*. This was the first step towards different types of integration with incredible results.

In 1831, Cauchy had discovered another incredible formula known as the *Cauchy Integral Formula*. He first published this discovery on October 11, 1831. However, the formula didn’t become available to use until 1841. The formula is . This formula made it possible to integrate functions within circular regions and if z0 is not within the circular region then . So this is an amazing formula that makes a complicated function much easier to integrate.

The next contribution to *Complex Analysis* came from Joseph Liouville who was a French mathematician and a professor at the College de France. In 1847, a German mathematician by the name of Carl Wilhelm Borchardt had sat in on Liouville’s lectures and had published them in 1879 and named theorem of Liouville’s after him. This became known as *Liouvile’s Theorem* which states that *every bounded entire function is constant*. More formally its stated as *let f be a holomorphic function in the* *complex plane. Suppose there exists some real number* *such that* *for all z. Then f is a constant function.* This was a huge discovery that made analyzing complex functions easier. However, it was actually Cauchy who had first derived this theorem in 1844.

Next, came P.A. Laurent who served in the army as an engineer. Laurent had shown that the representation of *Cauchy’s Theorem* could be of power series of holomorphic functions in discs where negative exponents of appear are allowed. However, Laurent’s original work was never published. In 1863, long after he had passed away his wife had gotten his proof of his work published which was a very tedious proof. Laurent also used in his proof the Cauchy integral method. This became known as *Laurent series* in the complex plane which became a very important result for *Complex Analysis*.

One of the most important theorems in *Complex Analysis* is the *Residue Theorem*. It all started when Cauchy first started to look into function theory. The goal of this theorem was to find a way to calculate definite integrals by going from the real plane to the complex plane. This theorem involved *Laurent series*. The *Residue Theorem* was used to integrate complicated rational functions by which who find the *poles* of the function. *Poles* are found by setting the expressions in the denominator equal to zero and then solving for . Then for each value of you cover up the expression that will make it equal zero so that it’s not undefined and then plug it into the other expressions in the denominator. This keeps getting done until all the *poles* are evaluated. This theorem is the biggest breakthrough in *Complex Analysis*.

Overall, complex numbers and *Complex Analysis* have many applications in the real world such as electrical engineering, quantum mechanics, and many more.

**Bibliography**

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