David Hilbert: Lifting the Veil

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May 1, 2017

Young David Hilbert

In the small town of Königsberg, on January 23, 1862, David Hilbert was born to Otto and Maria Hilbert. The firstborn child and only son would one day become the most influential mathematician in Germany, and quite possibly in the entire world. His hometown Königsberg, the capital of Prussia, was a small town which was also the home of philosopher Immanuel Kant. He was raised rather strictly, particularly by his father who was a very rigid man. It is likely that his mother homeschooled him through the early years of his childhood, as evidenced by the fact that he did not start school until the age of eight, two years later than most of his peers.

At this time, his parents enrolled him in what was known to be the best school in Königsberg, the Friedrichskolleg. Despite its reputation as the best school in town, it was far from a nice fit for the young Hilbert, especially considering the mathematician he would one day become. The main focus of the education was on Latin and Greek, with not much attention paid to mathematics. When his parents noticed that David was not having as much success at the Friedrichskolleg as they had hoped, they removed him from this school and moved him into Wilhelms Gymnasium, where he would spend the last year of his schooling. At both schools it became clear that Hilbert was a talented mathematician, but he was certainly no prodigy. It was still not clear exactly how talented he was. This was due in large part to the fact that Hilbert was not the greatest student, and that he did not focus much of his efforts on school mathematics, knowing that he would one day turn back and spend much time on the subject.

At the age of eighteen, Hilbert enrolled in the University of Königsberg, a university which was much smaller and much less reputable than the University of Berlin. However, Hilbert was not always much of a traveler, and in fact did not spend much time outside of his university during his years as a student. The University of Königsberg had a strong tradition in mathematics and physics, and had been the school of Carl Jacobi and physicist Franz Neumann. Heinrich Weber would hold the chair in mathematics from 1875 to 1883 before being succeeded by Hilbert's advisor, Ferdinand Lindemann. While at the university, Hilbert had the opportunity to take lecture courses from Weber himself, in topics including elliptic functions, number theory, and invariant theory. This was perhaps Hilbert's first push into the field of invariant theory, and along with some encouragement from his advisor, would cause him to devote much of his future study to this field.

Arguably more important in Hilbert's lifetime than these professors under whom he studied, were the friendships he made with two other mathematicians at the school. His friendship with Hermann Minkowski, and especially with Adolf Hurwitz, would shape the path of Hilbert's studies and would have a strong impact on the way he thought about mathematics. Minkowski was somewhat of a prodigy (unlike Hilbert) – despite his being two years younger than Hilbert, he was a semester ahead of him at the university. He spent much more time travelling during his years as a student, studying also at the University of Berlin, which allowed him to learn from the great mathematicians Kronecker, Kummer, and Weierstrass. He would soon take what he learned back with him to Königsberg, where much of this was shared with Hilbert. Perhaps even more influential was Hurwitz, who, despite being only three years older than Hilbert, was a professor at Königsberg. Hurwitz, formerly a student of the great Felix Klein, went on frequent walks with Hilbert for the sole purpose of discussing mathematics. It was Hurwitz who ultimately inspired Hilbert to become a universal mathematician. As mentioned, Hilbert studied invariant theory under his advisor, Lindemann. His doctoral thesis, titled *Über invariante Eigenschaften specieller binärer Formen, insbesondere der Kugelfunctionen* (*On invariant properties more specially binary forms, especially the spherical harmonics*) was clearly focused on this subject. While this was not an especially poor thesis, it also was not enough to give Hilbert the reputation that he would eventually have among mathematicians.

Early Career and Work in Invariant Theory

Upon graduation, Hilbert travelled to Leipzig where he would meet Klein for the first time, and later to Paris where he met Henri Poincaré. Poincaré and Hilbert would eventually lead their respective nations in mathematics, but as we will see they were not able to establish a strong friendship. The two had very different styles, as Poincaré was more likely to leave a lot of work to be done by the reader, while Hilbert would fill in all of the gaps in his work. Similarly, Poincaré's work was much more geometric, while Hilbert's was much more algebraic. Another potential reason for their poor relationship is that Poincaré was likely not very personable, as evidenced by the fact that he had few followers and few students despite his great success (Hilbert, on the other hand, would take 69 students in his career).

Hilbert would soon return to Königsberg, where he worked as a *Privatdozent*. This means that he worked as a teacher for the university, but was unpaid. During this time he worked toward completing his *Habilitation*, which would give him the right to teach full-time at a German university. This was completed rather quickly and, unsurprisingly, he elected to teach at Königsberg, a school which he liked very much and which was also home to his great friend Hurwitz. At this point, Hilbert still had not produced any work that was sufficient to establish him as the mathematician he would become.

Hilbert's emergence onto the scene in the mathematical community came largely as a result of his visit to Erlangen, where he met Paul Gordan, the "king of invariant theory." After spending only a week away, Hilbert had already finished the first of many papers which he would eventually write on the topic of invariant theory. His breakthrough came in a two-page paper, in which he proved the existence of a finite basis for a certain set of objects. Interestingly enough, this was a proof by induction and was not a constructive proof; that is, while proving that such a basis must exist, he made no claims about what that basis might be. This non-constructive existence proof was not very common at the time, and Gordan, desiring to know the in- and covariants and the relationships between them, complained that "This is not mathematics, it is theology." Kronecker also expressed trouble with Hilbert's general existence proofs. He insisted on finite algorithmic procedures, which was in direct opposition with Hilbert's tendency towards more abstract methods, which was probably brought about by this early success in invariant theory. Despite the backlash from two important mathematicians of the day, Hilbert was confident that he had provided a successful proof. At last he had established himself as a research mathematician.

By this point a bit of a rivalry had formed between the Universities of Berlin and of Göttingen. Klein, who himself was from Göttingen, saw Hilbert's potential early on, probably as early as their first meeting in Leipzig. As a result, Klein pushed to have Hilbert's works published in the

Mathematische Annalen, a journal based in Göttingen. Hilbert's first major work, already discussed, was published in the *Annalen,* as was much of his subsequent work. This was beneficial for the journal, for Klein, and for Hilbert.

Hilbert as a Number Theorist

In 1892, Hilbert married his second cousin Käthe Jerosch. Nine months later Käthe would give birth to their only child, Franz. Around the same time, in 1892, Hilbert surprisingly left the field of invariant theory altogether, claiming essentially that all of the most important tasks in invariant theory had already been resolved. He began studying number theory, which he had only a little experience in. This type of drastic shift was not very common for mathematicians, but Hilbert certainly expressed a great deal of confidence in doing so, claiming that he would be able to become as relevant a number theorist as he was an invariant theorist. This claim would turn out to be correct, and his success began with his first work in number theory, a simplified proof that both π and e were transcendental; that is, neither could be the root of a polynomial with rational coefficients. Within a few years he moved to Göttingen, where he was offered a position as the chair of mathematics at the university. Here he taught for the rest of his career, partnering with Klein. With this move, the rise of Göttingen's mathematical prestige was underway.

Before his move to Göttingen, Hilbert was asked, along with his good friend Minkowski, to give a report on the status of the field of number theory. The two were able to publish their *Zahlbericht*, or *Report*, by 1897, and that with much success. There were in fact two major successes of this *Report.* The first is that it simultaneously served the purpose of reporting and synthesizing the work of Kronecker, Kummer and Weierstrass as well as Hilbert's own work. It also presented a history of the field. Bringing all of this together effectively allowed mathematicians new to the topic of number theory to start with the *Report,* and thus the need to search for and read through a lot of older literature was eliminated. Although number theory was not always highly-regarded by many of the leading mathematicians and physicists, Hilbert and Minkowski gathered some momentum for the field by arguing that many of these seemingly unimportant problems actually could have far-reaching effects in other fields of mathematics.

Understanding the Foundations of Geometry

While this *Report* was a huge success, it was essentially Hilbert's only noteworthy one in the field of number theory. After spending less than ten years on the subject, he made yet another shift in his focus and began his study of geometry. With this, Hilbert, who had now made himself relevant in two independent fields of mathematics, embarked on achieving the same task in a third. He turned out to be quite successful in this, as his study of geometry led him to some of the works for which he is most remembered.

In particular, Hilbert is known for his analysis of the axiomatization of geometry. He worked to have the entire field built up out of the logical structure which follows only from these axioms. Towards this goal, he wanted to make sure that the truth of geometric statements was based only on this logical structure, and not on any meanings or ideas which came before ever thinking about geometry formally; in other words, the results should follow from the definitions and axioms, not from inherent ideas about points, lines, and so on. In the words of Hilbert himself, "One should always be able to say, instead of 'points, lines, and planes', 'tables, chairs, and beer mugs'" (Gray, p. 49). Hilbert's famous *Grundlagen der Geometrie* (*Foundations of Geometry*), published in 1899, was a major work in this analysis of the axioms of Euclidean geometry. With this, he was able to make major improvements even on the achievements of the Greeks in the axiomatization of this geometry. His intention with this work was to show that the axioms are both independent and consistent. Proving independence equates to proving that none of the axioms follow logically from the rest, hence one cannot make the set of axioms any smaller by simply removing one (without changing the geometry altogether). On the other hand, proving consistency equates to showing that this set of axioms will not create any logical contradictions. Hilbert found success in proving the consistency of the axioms, so long as the axioms of arithmetic are themselves consistent, but he was not able to prove entirely that the axioms are independent. The natural question following Hilbert's conclusion about the consistency of these axioms is clear, and is one that Hilbert himself would focus on quite a bit in the years to come. The question is: are the axioms of arithmetic consistent as well? A proof that this is the case would complete the corresponding proof in geometry and would provide a lot of clarity.

Lecture in Paris

With the turn of the century came some of Hilbert's most memorable work. He was asked to give a lecture at the Second International Congress of Mathematicians, which would be held in the August of 1900 in Paris. Hilbert, understanding his own reputation and the way he was regarded by the mathematical community, knew that the topics he chose to discuss here would receive a lot of attention by some of the brightest minds. With this in mind, Hilbert spent nearly a year preparing for his lecture. He thought about topics that interested him, topics that were interesting or challenging for the entirety of the mathematical community, and issues regarding the relationships of fields within mathematics with each other as well as with their applications, especially in physics. In September of 1899, Hilbert listened to a lecture given by physicist Ludwig Boltzmann, which was focused more on the future of physics than on the past. It is likely that this lecture is what provided Hilbert with much of his inspiration to begin thinking in terms of the future of mathematics, rather than the past.

Hilbert's lecture would be motivated by a handful of big ideas that he had about mathematics as a whole. First, he was a man of problems. Hilbert believed that problems were central to mathematics, and that good problems did not disappear when solved, but rather led to other problems or even to entirely new fields of mathematics. Second, he saw it both necessary and possible to break the divide which had slowly formed between pure and applied mathematics. While many people (even within the mathematics and physics communities) were beginning to see pure mathematics as a study of "math for math's sake," with no important applications in physics, Hilbert would claim that "Problems can have the most unexpected significance" (Gray, p. 4). On a similar note, Hilbert believed that mathematics can and should be seen not as a collection of different ideas but as a coherent whole. He therefore sought to unify mathematics and to apply everything, particularly to physics. Third, Hilbert was an optimist in an era of pessimists. He believed that every problem could be solved – that there is nothing that we cannot know – and he wanted all mathematicians to subscribe to this idea. Finally, Hilbert was influenced by his discussions with Minkowski and Hurwitz. Minkowski further encouraged Hilbert to look to the future, proposing that possibly people would continue talking about this lecture for decades if Hilbert took this piece of advice. Minkowski could not have known at the time the scope of this lecture, and could not have imagined that the "decades" of discussion about this lecture would be an underestimate. With these big ideas gathered, and a list of problems at hand, David Hilbert was ready to set the stage for twentieth century mathematics. He began:

> *Who among us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries? What particular goals will there be toward which the leading mathematical spirits of coming generations will strive? What new methods and new facts will the new centuries disclose in the wide and rich field of mathematical thought?*

Hilbert would go on in this magnificent lecture to do exactly that: to lift the veil which separated the present from the future. Not only did Hilbert promote many of the big ideas discussed above, but he also presented a list of 23 problems that were to guide the study of mathematics in the 20^{th} century. Minkowksi would later remark quite accurately that Hilbert would be acknowledged in mathematics as the general director in the years to come.

Hilbert's Problems, as they would be called, touched on areas of number theory, geometry, analysis, algebra, and more. Hilbert also included many problems which were not of particular interest to him in his research but were widely regarded as some of the important problems of the day. For example, among these problems were listed the continuum hypothesis, the Riemann hypothesis, Goldbach's conjecture, and more. Hilbert, though he would eventually publish this lecture with a full list of the 23 problems, only spoke about ten of these in his lecture in Paris. Many of these have been resolved in the 117 years since the lecture, but many have not. Of those left unsolved, some have been shown to be impossible to prove or disprove (which can be considered in some sense a solution to the problem), while others have even more uncertainty still surrounding them.

Of Hilbert's 23 Problems, about three quarters of them were problems he had tried to solve or were directly related to fields that he had studied. This is noteworthy for two reasons: first, it was wise for Hilbert to focus on the areas that he knew most about and to limit the number of problems, rather than making a much longer list; second, this number is remarkable considering the wide range of topics which the problems cover. Hilbert touched on each of the three most general fields of mathematics at the time – those being analysis, algebra, and geometry (while all other fields or subfields somehow fit into one or multiple of these). He also addressed issues regarding the applications in physics, as evidenced by his second Problem, which will be stated and discussed later. Another strength of this lecture was Hilbert's emphasis on the need for rigorous general theories. Again, the finer subjects were not all touched upon by Hilbert in his Problems, as it was more beneficial to select a smaller number of Problems with which he was more familiar. Some noteworthy topics of the time which Hilbert did not address include the integration of trigonometric sums, problems in topology, problems in the theory of functions of several variables, and more.

A Brief History of Some Problems

The first of Hilbert's Problems relates to the continuum hypothesis, a problem originally posed by the great Georg Cantor. The hypothesis is that there is no set whose cardinality lies strictly between that of the set of natural numbers and that of the set of real numbers. Cantor was wholeheartedly convinced that this is true, and spent years trying to prove that this was the case. With no proof being yet discovered by 1900, Hilbert considered it worthy of being his first Problem. In his lecture he spoke of the notion of well-ordered sets, and even suggested that this may somehow be the key to proving the continuum hypothesis. This problem was partially resolved forty years later, when the great logician Kurt Gödel proved that the continuum hypothesis could not be disproved within the current axiomatic system, Zermelo-Frankel set theory (ZF), even with the stronger hypothesis of ZFC, which includes the axiom of choice. The rest of the solution did not come until 1963, when Paul Cohen proved that the continuum hypothesis could not be proved within this system either. Together, these discoveries showed that the continuum hypothesis is actually completely independent of ZFC axioms; that is, either the continuum hypothesis or its negation could be accepted as an axiom without causing any inconsistencies.

Hilbert's second Problem calls for a proof of the consistency of the axioms of arithmetic. Recall that he had already shown that the axioms of Euclidean geometry are consistent so long as the axioms of arithmetic are. With that in mind, there was a lot resting on this problem. With this question, though, Hilbert found himself in the middle of a number of philosophical discussions. One example of such a discussion was based around the idea of axiom systems being complete - that a system of objects obeying certain axioms must exist, and that there is no larger system of objects satisfying the axioms. Gottlob Frege argued that completeness axioms cannot be used to resolve questions of existence. At present it is not generally agreed upon whether this particular Problem has been resolved or not. In 1931, Kurt Gödel showed with his second incompleteness theorem that no proof of the consistency of the axioms can be carried out within arithmetic. In 1936, Gerhard Gentzen proved that Peano Arithmetic (which is the standard) is consistent and that the proof can be obtained in a system which is weaker than set theory. The mathematical community is undecided about whether these provide a solution to the Problem as stated by Hilbert.

The third Problem is the first one we encounter which has a definite solution. It is one of the many Problems from geometry. There was a sort of cut-and-paste proof at the time that the area of a triangle with a given height is proportional to the length of the base. The cut-and-paste proof consisted of making a copy of the triangle and pasting the copy next to the original in such a way that a parallelogram was formed. Although a similar result had been proven in three dimensions, the proof required calculus. In this Problem, Hilbert asks for a proof that a cut-and-paste argument cannot be used to prove this corresponding result in three-dimensions, and that the use of calculus in the proof was therefore necessary. In his lecture, Hilbert states the Problem as asking for “two tetrahedra of equal bases and equal altitudes which can in no way be split up into two congruent tetrahedra, and which cannot be combined with congruent tetrahedra to form two polyhedra which themselves could be split up into congruent tetrahedra” (Gray, p. 252). Hilbert’s student Max Dehn successfully proved the statement within a year. He did so by finding a quantity involving side lengths and dihedral angle measures which was invariant under such copy-and-paste moves. This Dehn invariant was essential to showing that such an argument cannot be used in three-dimensions.

Hilbert’s sixth Problem is one in which we see his interest in the applications of mathematics, particularly in physics. It asks for the axioms of physics to be treated with more mathematical rigor. This Problem, though it was not stated specifically enough to decide whether it has been resolved or not, was certainly influential in some ways in the work of mathematicians and physicists alike throughout the 20^{th} century. Another example of his desire to connect pure mathematics to physics is Problem 23, which focuses on developing the calculus of variations. An example of this is the question of the curve of quickest descent, which is relevant in physics for seeing the path through which light will travel in certain media (depending on whether the medium is homogeneous, a variable which is in some sense discrete, or a variable which is continuous). In providing Problems such as these, Hilbert successfully gathered much interest in a subject in which mathematicians of the time were typically not very interested.

Hilbert’s seventh Problem is an example of one from the field of algebraic number theory. It concerns the transcendence of certain numbers. In particular, Hilbert claims and requests a proof that numbers of the form a^b, with a an algebraic number and b an irrational algebraic number, is always transcendental or at least irrational. The question is also asked in an equivalent form in geometric terms, relating properties of an isosceles triangle. In 1934, Aleksandr Gelfond provided the proof which Hilbert sought, and the Problem was thus resolved.

Hilbert’s eighth Problem deals with the distribution of prime numbers and is one of particular interest. Among these problems are the Riemann hypothesis, Goldbach’s conjecture, and the twin prime conjecture, none of which have been proven to date. Goldbach’s conjecture states that every even integer larger than two can be written as the sum of two primes. This has not yet been solved, though a close result – that every sufficiently large odd number can be written as the sum of three primes – was proved by Ivan Matveyevich Vinogradov in the 1930s. The twin prime conjecture states that there are infinitely many twin primes (pairs of primes which are only two apart, such as 17 and 19). This has not been proven yet either. The Riemann hypothesis “has been the Holy Grail of mathematics for a century and a half” (Borwein, p. 3). Hilbert himself once said, “If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?” As it turns out, Hilbert may have been wondering about exactly the right question, as this Problem has stumped mathematicians thus far. The claim is that the nontrivial zeroes of the Riemann zeta function (a complex function) have real part equal to ½. This, too, has a great deal to do with the distribution of primes. For each of these three claims about primes there is overwhelming numerical evidence in favor of their truth, however none has been proved generally. In particular, Goldbach’s conjecture holds for up to $4*10^{18}$, twin primes have been found as large as $\sim10^{388,000}$, and the Riemann hypothesis has been checked for the first ten trillion zeroes. These problems, among others listed by Hilbert, remain open today.

Hilbert's twelfth Problem was one that he cared about particularly. It asked for the extension of Kronecker's theorem on Abelian fields to any algebraic realm of rationality. This Problem is significant because it is an example of Hilbert's dream of unifying mathematics. In this Problem there are hints of at least the fields of algebra and number theory, and it is evident that Hilbert wanted to bring not only these subjects but all of mathematics together to be seen as a coherent whole. This Problem is also interesting because much of what Hilbert claimed was actually incorrect, which caused confusion among mathematicians in this area for many years.

The following set of six Problems were not particularly influential in the years to come, especially compared to many of the others. Problem 14, however, was the only of the 23 Problems which dealt directly with Hilbert's first area of expertise, invariant theory.

For obvious reasons, every time that one of these Problems was solved, it was a noteworthy event in the history of mathematics. The question remains whether those Problems which are still open will ever be resolved, and if so, how that might happen. Hilbert did not work on all of these Problems himself, and those to which he gave some attention did not receive very much of it. He did, however, have the opportunity to watch the mathematical community change before his very eyes, in many of the ways about which he had once dreamed, as the great mathematicians went to work on these Problems and similar ones. At least seven of his 23 Problems had been at least partially resolved by the time of his death.

Hilbert's Space Theory Marks the End of His Career

After this climax of Hilbert's career, he made yet another shift in his research. Taking on a fourth field, Hilbert discovered what we now call Hilbert space theory. Though named after Hilbert, this space theory is due in many ways as much to the work of some of his students – particularly Ernst Hellinger, Hermann Weyl, and Alfred Haar - as it is to Hilbert. Other students of his – he had 69 in total over the course of his career – worked on some of the Hilbert Problems, but many were much more interested in Hilbert space theory, particularly because that was the subject which had most of Hilbert's attention at the time.

In 1930, Hilbert retired from his position at Göttingen, though he continued to give lectures there until 1933. Hilbert received a number of honors towards the end of his life, including his election as an honorary member of both the London mathematical society and the German mathematical society. In 1942, after falling and breaking his arm, Hilbert's health began to decline until he eventually died in the next year. David Hilbert, the man who unveiled the future of mathematics at the start of the twentieth century and promoted optimism and enthusiasm about problem-solving in mathematics, leaves us with the following words of encouragement, spoken in 1930: "We must know, we shall know."

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