Complex Numbers: Interpretations and Developments

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Introduction

The history of the study of complex numbers endured much evolution and controversy from some of the most renowned mathematicians of the time. Some great mathematicians, like Newton, completely turned down the notion of a complex number, while others were suspicious of their existence. In fact, it was only during the 16th century that mathematicians even began to study these so-called imaginary numbers [1]. Before then, if a mathematician arrived at an answer with the square root of a negative number, he would automatically discard that solution [1]. The focus of this paper is on the various developments and interpretations on the square root of a negative number put forth by Hieronimo Cardano (1501-1576), Rafael Bombelli (1526-1572), Gottfried Wilhelm Leibniz (1646-1716), René Descartes (1596-1650), John Wallis (1612-1703), Caspar Wessel (1745-1818), Robert Argand (1768-1822), Abbé Adrien-Quentin Buée (1748-1826), William Rowan Hamilton (1805-1852), and Leonhard Euler (1707-1783). The paper will conclude with a brief excerpt about complex function theory and some of the most important theorems related to complex numbers, including *Euler's identity*, the *Cauchy-Riemann Equations*, and *Laplace's equations*.

The Beginnings

The first known problem involving complex numbers arose in the 1st century AD when the Greek mathematician Heron of Alexandria (10 AD – 70 AD) attempted to calculate the volume of a frustum of a pyramid [1]. In one part of his solution he arrived at the quantity $\sqrt{81-144}$, but he immediately threw aside this notion of the square root of a negative number. Mathematicians from then on followed in his footsteps, and any thoughts about the square root of a negative number were tossed aside. In 486 AD we find the Indian mathematician Bhaskara Acharya who claimed that the square root of a negative number does not exist since negative numbers cannot be a square [1]. The thoughts of Heron and Acharya seemed to be the trend for the next eleven centuries, for whenever mathematicians came across the square root of a negative number in their answer, like Heron, they disregarded these solutions, for they thought these types of solutions were not mathematically possible and thus were meaningless. It was not until the 16th century that mathematicians started to ponder about what these concepts actually meant.

Introduction to the Study of Complex Numbers

In the 16th century we find the first person to think about the meaning of a complex number: Cardano (1501-1576). In 1545, he published his *Ars Magna*, which contains one of the most historic achievements in mathematics of the time discovered by Scipio Del Ferro (1465-1526): the general solution to a cubic equation on the form $x^3 + px = q$ [5]. He attempted to solve the cubic equation $x^3 = 15x + 4$ for x, for which he found that $x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$. He, like great mathematicians before him, claimed there was no solution, and he asserted a solution of this type was ridiculous, but nonetheless, he was not "afraid" of these types of solutions like others previously [4]. Although he thought this type of solution was bizarre, this was a historically significant result because it represented the first time that the square root of a negative number was written down [4]. Another turning point in the beginning of the study of complex numbers was when Cardano solved for x and y in $x + y = 10$ and $xy = 40$ [3], finding that $x = 5 + \sqrt{-15}$ and $y = 5 - \sqrt{-15}$. After multiplying these solutions together, he concluded that the result was 40, and he was confused as to how solutions containing the square root of a negative number could represent a real number. It was Bombelli (1526-1572) who elaborated on this idea and proposed his interpretation of complex numbers [4].

Bombelli considered the same cubic equation as Cardano, $x^3 = 15x + 4$ [4], and by inspection he found that $x = 4$ was a solution [4]. After long division and factoring, he found that $x = -2 + \sqrt{3}$ and $x = -2 - \sqrt{3}$ were the other two solutions [4]. Recall that solving this cubic using Cardano's method yields a solution of $x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$, seemingly different than Bombelli's solutions. The curious type of person would ask themselves what happened to Cardano's solutions, and what is exactly what Bombelli did. He noted that the two terms of Cardano's solution only differed in sign, so he denoted $\sqrt[3]{2 + \sqrt{-121}}$ as $a + \sqrt{-b}$ and $\sqrt[3]{2 - \sqrt{-121}}$ as $a - \sqrt{-b}$ [4], and after much manipulation of these two expressions, he eventually concluded that the sum of the two is 4. With this Bombelli had given his interpretation that the two complex numbers Cardano arrived at were in fact real, but they were denoted in an unfamiliar notion.

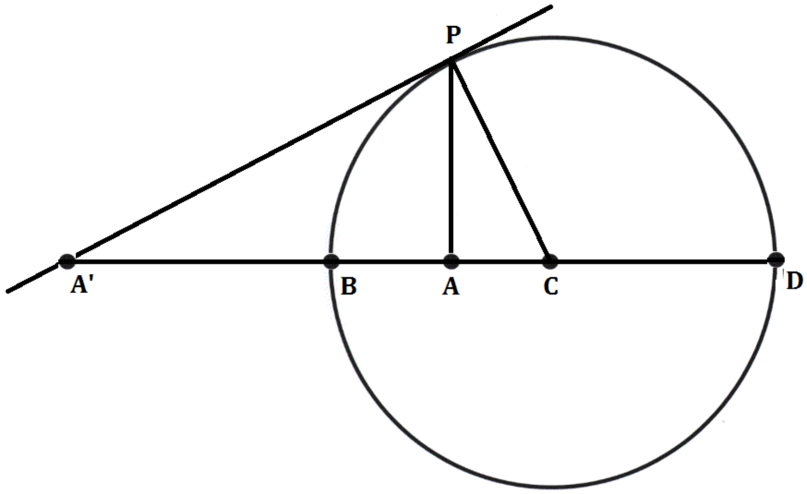
This discovery revolutionized the mathematical world, for he had interpreted what a complex number meant, therefore creating the study of complex numbers. Bombelli published his results in *L'Algebra*, which contained all his notions about complex numbers [4]. Some mathematicians developed on Bombelli's ideas, yet some were skeptical of this radical idea and thus were not satisfied with his arguments. The movement to dig deeper into the meaning of complex numbers did not end here, though. In fact, Bombelli's discovery was only the beginning of a new branch of mathematics.

A Geometrical Interpretation

About 100 years later, many mathematicians accepted Bombelli's contributions, but the great Leibniz still was not convinced. Leibniz (1646-1716) was not satisfied with Bombelli's treatment of Cardano's formula, and Leibniz did not comprehend how adding two complex numbers would yield a real number [4].

In the 17th century we find that mathematicians were not satisfied with just a general meaning of complex numbers. Mathematicians were baffled at what the geometric interpretation could be, and many played their hand at deducing such an interpretation. Descartes (1596-1650) was the first mathematician to try to find a geometric interpretation for imaginary numbers, but after much analysis, he concluded that imaginary numbers could not be associated with a geometric construction [4].

The next person to attempt to construct a geometric interpretation of imaginary numbers was Wallis (1612-1703) [4]. The key word here is *attempt* – Wallis was able to deduce an argument of some sort, but his argument was not quite convincing enough for the mathematicians of the time, and he had nothing revolutionary to say about $\sqrt{-1}$. He composed a complicated geometrical argument, which concluded with his construction of a diagram of this sort below and claiming that $A'P = \sqrt{(BA')(A'D)}$ [4].

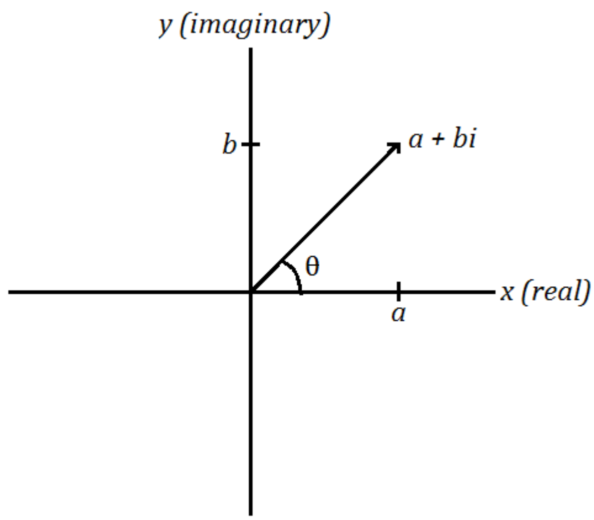


Wallis' construction

His argument that he had come up with some sort of interpretation for $\sqrt{-1}$ was based on his thought that BA' was a negative distance while $A'D$ was positive. He believed that because a negative number can be thought of as starting at 0 on a number line and moving left, then BA' was a negative distance [4]. He therefore concluded that $(BA')(A'D)$ was negative and thus $A'P$ represented the square root of a negative number.

Once we enter into the 18th century, we find that people could not wrap their heads around a geometrical, or more generally, a *logical*, explanation of complex numbers. The notion of complex numbers even seemed to break all rational explanations of mathematics. For a, b negative integers, is $\sqrt{ab} = \sqrt{a}\sqrt{b}$? Well, with complex numbers, an example of where this is not true is $70 = \sqrt{(-100)(-49)} = \sqrt{-100}\sqrt{-49} = (10i)(7i) = (70)(-1) = -70$. The search for a geometric interpretation of $\sqrt{-1}$ was not over, though, despite the seemingly illogical nature of $\sqrt{-1}$. We find that Wessel (1745-1818), ironically not even a mathematician, but rather a surveyor, was able to crack the mystery behind $\sqrt{-1}$.

Wessel presented his paper (1797) that simplified the geometric interpretations of complex numbers that Wallis originally suggested [4]. He considered the standard notion of the real number line, where starting at 0, moving to the right suggests the numbers are getting larger, and moving to the left suggests the numbers are getting smaller, thus becoming negative. He built upon this, creating a vertical "imaginary" axis perpendicular to the horizontal "real" axis, therefore creating the *complex plane*. A complex number $a + bi$ represented moving a units right (if a is positive) or $|a|$ units left (if a is negative) and b units up (if b is positive) or $|b|$ units down (if b is negative), and Wessel denoted this notation the *rectangular*, or *Cartesian*, form [4]. He built upon this idea, later creating the *polar* form, which is as follows. He would denote $a + bi$ as a vector, and he let θ represent the angle formed from the positive part of the horizontal axis to the vector (a counterclockwise motion). Using the Pythagorean Theorem and a basic knowledge of trigonometry, he concluded that $a + bi = \sqrt{a^2 + b^2}\,(\cos(\theta) + i\sin(\theta))$, writing that $\sqrt{a^2 + b^2}$ is the *modulus* of the complex number $+bi$ and θ is the *argument* of the complex number [4].



Wessel's construction: the *complex plane*

Wessel was historically significant because he was the first to explicitly denote that the imaginary axis is perpendicular to the real axis, even though some mathematicians such as Henri Dominique Truel and Karl Friedrich Gauss (1777-1855) proposed the idea before but never published their results [4]. (Gauss was monumental in the sense that he was the first person to use the phrase *complex number* [2].) Another monumental discovery which stems from Wessel's definition of the complex plane was that i represents the vector with no movement on the real axis and 1 unit up on the imaginary axis [4], to which the reader can see that this represents a 90 degree rotation counterclockwise. More generally, if we multiply an arbitrary vector $a + bi$ by i, we find that the product $(a + bi)i = ai + bi^2 = ia - b = -b + ai$. Essentially, $\sqrt{-1}$ can be thought of as a rotation operator by 90 degrees [4].

Even though Wessel introduced and published quite a few revolutionary arguments, many mathematicians were still not convinced of his discoveries. However, the two mathematicians Argand (1768-1822) and Buée (1748-1826) rediscovered and supported his ideas [4]. Argand's approach (1806) is quite similar to Wessel's argument and will not be repeated here [2]. In fact, Argand is quite often mistakenly credited as the first person to offer the geometric approach that Wessel made because his ideas did not receive as much fame as Argand's. Buée's argument, however, is different from Wessel's but still results in his same conclusions.

Buée's original goal was to answer the questions posed in Lazare Carnot's (1753-1823) publication *Géométrie de Position* (1803) [4]. Carnot considered the division of a line segment of length a into two segments such that when the two lengths are multiplied together, the product is equal to $\frac{1}{2}$ the original side squared. He attempted to solve the equation $x(a-x)=\frac{1}{2}a^2$ for x, of which the answer is found to be $x=\frac{1}{2}a \pm (i)(\frac{1}{2}a)$. Like any mathematician before Cardano, Carnot was puzzled by the $\sqrt{-1}$ in his answer and thus assumed his initial condition was not possible. Buée, however, challenged this and offered a response. He claimed that the imaginary part of the answer, $\frac{1}{2}a$, represents that the point, which would divide a into two segments according to the original condition, is located $\frac{1}{2}a$ units away perpendicular to the line segment. He did not justify his claim, though, making him lose credibility in the eyes of some mathematicians of the time.

Looking back at the 17th and 18th centuries, it is clear that Wessel was the winner in terms of a geometrical argument that many mathematicians seemed to be fond of. Because of Wessel, mathematicians were able to find the geometrical approach they were longing for. As we enter the 19th century, we find some mathematicians dissatisfied with the approaches put forth, and some looking to further expand upon the theory of complex numbers.

An Algebraic Approach

As we enter the 19th century, we find Hamilton (1805-1852) unhappy with the geometric interpretation of $\sqrt{-1}$; in fact, he did not even think $\sqrt{-1}$ should have a geometric interpretation, for he believed there should only be an algebraic interpretation [4]. In his publication *Theory of Conjugate Functions or Algebraic Couples: with a Preliminary Essay on Algebra as a Science of Pure Time* (1835), he considered what he called an *ordered pair (couple)* of real numbers (a,b) [4]. We can see that Hamilton based this notation off of Wessel's notation of $a+bi$. Hamilton defined addition as $(a,b)+(c,d)=(a+c,b+d)$ and multiplication as $(a,b)(c,d)=(ac-bd,bc+ad)$ [4].

Start of the Theory of Complex Functions and Some Monumental Theorems

In the 19th century it seemed as though that mathematicians had learned the basics of complex numbers. Eventually, they began to consider functions of complex variables. Denoting $z=x+yi$ as our complex variable, they denoted the complex function to be $f(z)=f(x+yi)$. Another way to write this is $f(z)=u(x,y)+iv(x,y)$, which will come back into play in the "Cauchy-Riemann Equations and Lagrange's Equations" section.

Euler and Euler's Identity

Although Wessel is generally credited with many monumental contributions to $\sqrt{-1}$, Euler (1707-1783) knew the notions behind these contributions before Wessel [4]. In 1748, when Wessel was just three years old, Euler published the identity $e^{\pm ix}=\cos(x)\pm i\sin(x)$. Substituting in $x=-\frac{\pi}{2}$ in for $e^{\pm ix}=\cos(x)\pm i\sin(x)$, we arrive at $e^{-\frac{\pi}{2}}=i^i$. What we have just shown is that an imaginary number raised to an imaginary number is a real number, what mathematicians including Argand once thought was not possible [4]. Additionally, substituting in $x=\pi$, we arrive at *Euler's identity*: $e^{i\pi}+1=0$. Some call this the most beautiful theorem in all of mathematics, as it so elegantly combines five of the most fascinating numbers [4]. Furthermore, not only is it such a beautiful theorem, it holds many practical applications in differential equations and engineering.

Cauchy-Riemann Equations and Lagrange's Equations

The works of Augustin-Louis Cauchy (1789-1857) and Bernhard Riemann (1826-1866) paved their way for the start of the theory of complex functions [2]. Let us revisit $f(z) = u(x,y) + iv(x,y)$. In order to study the theory of complex functions, one must be able to understand how to find $f'(z)$. The Cauchy-Riemann Equations so elegantly define $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ as $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ [4]. These equations form the basis for complex function theory [2], and taking the derivative of each of the equations yields $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}$, $\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$, and $\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}$.

By combining $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}$ and $\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}$, one gets $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$, and by combining $\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$, one gets $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. These two equations are known as *Laplace's equations* [4], which have immense applications in engineering.

Conclusion

The study of the developments and interpretations of complex numbers has had quite an arduous history. From considering what the square root of a negative number even is to delving into the theory of complex functions, there is so much to learn if trying to have a comprehensive understanding on all these topics, and we have Bombelli, Wessel, Hamilton, Euler, and more to thank for their revolutionary contributions to this field.

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