Lynne Richman
History of Mathematics
Professor Zeilberger
Final Project

The History and Evolution of Numbers

Numbers are essential to mathematics. We use them to describe, to measure, to predict, to experiment, and so much more. It is nearly impossible to imagine a world without numbers and certainly it is impossible to imagine mathematics without numbers. However, the spectrum of numbers that we recognize and use in this modern era has not always been as broad and inclusive as it is today. Throughout history, various problems have arisen that forced the acceptance of different types of numbers; such as rational, irrational, negative, and imaginary numbers; that we now include in our definition of a number with little to no hesitation.

At the beginning of recorded history, our ancestors used only what we now call natural numbers. These numbers were introduced as a matter of convenience and were used primarily for counting and ordering (Struik 10). The numbers of this ancient period were written in unary notation, with collections of sticks or pebbles often used to represent different quantities (Kelley). As civilization evolved further and the concept of commerce began to develop, our ancestors found themselves in need of representing larger numbers where unary notation would be cumbersome and inefficient. Thus, the notion of "bundling" became more common, with new symbols introduced to take the place of certain numbers like 5, 10, or 20 (presumably due to the fact that humans have five fingers) to shorten the notation of bigger numbers (Struik 11). This was the start of a new type of notational system for numbers, so that the unary system was gradually phased out of common use and replaced with quinary, decimal, and vigesimal systems depending on the civilization.

In these early numeral systems, position was not important. For example, with the ancient Egyptian decimal notation, you would simply add up the different "bundles" to arrive at the number that the string of symbols represented ("Egyptian numerals"). The order of the symbols carried no meaning for the number they represented (see Fig. 1 and Fig. 2). This lack of positional notation became problematic as our ancestors had to deal with increasingly larger numbers in their daily life. The first recorded instance of a positional number system was the Babylonian sexagesimal system ("Babylonian numerals"). In this system, when a number was written out as a string of symbols, the position of each symbol was important and carried meaning (see Fig. 3 and Fig. 4). The early Babylonians ran into problems when they needed to represent an empty position due to the fact that they had no symbol for zero. The use of a zero symbol (see Fig. 5) as a placeholder for such an empty position did not get introduced into the Babylonian number system until sometime between the fourth and first century BC (Geller).

This idea of having a symbol for zero as a placeholder was still a very long way off from approaching the concept of zero as a full-fledged number, like we know it today. As far as we know, the Indians were the first civilization to consider zero as a concept as opposed to just a symbol. One of the first mathematicians to use zero in computation was Brahmagupta, even going so far as to write rules for adding and subtracting with zero. The Arabians are credited with first writing the symbol for zero as an o, in the 9^{th} century AD. By the time of Descartes, in the 1600s, zero had become fairly common and generally accepted. People of this time period were fairly confident they knew how to add, subtract, and multiply with zero but division with zero was still fraught with abstract difficulties. The work of Newton and Leibniz dealt with this issue by introducing the concept of the infinitesimal as a way of tackling the obstacles that came with attempting division by zero (Wallin). By this time, zero was fully accepted into the definition of a number.

Travelling back to long before zero, the first set of numbers to augment the set of natural numbers initially used by civilizations was the set of rational numbers. The concept of fractions came somewhat naturally as society progressed and people found themselves dividing up surplus and other goods. One of the first recorded uses of fractions was by the Egyptians who expressed all rational numbers as the sum of unit fractions. Later, the Babylonians introduced sexagesimal fractions that did not have to be represented as the sum of unit fractions. However, the Babylonians did not have a convention or notation for differentiating between when a position should be read as an integer or fractional component of a number. For example, the same string of symbols would be used to represent $124=2*60^1 + 4*60^0$ and $2^1/_{15}=2*60^0+4*60^{-1}$. The Babylonians relied on the context to decipher the value of each position, which was certainly not the most efficient way of communicating ("Babylonian Numerals"). The Babylonian sexagesimal fractions were also limited in that they could only represent fractions in which the denominator was a factor of 60. The first civilization to use decimal fractions was the Chinese in the first century BC (Lay Yong 38) and the first recorded use of the horizontal fraction bar that we are so used to today was by al-Hassar in 1200 (Miller).

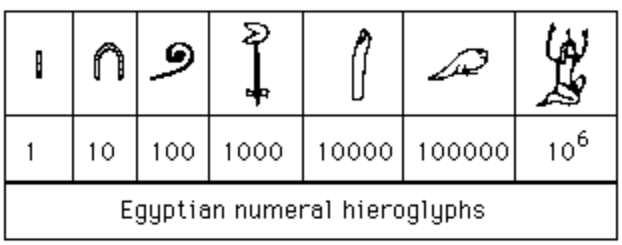
The next step forward in the development of a more complete notion of what a number can be was the concept of irrational numbers. The existence of an irrational number, specifically $\sqrt{2}$, is thought to have been first proved by the Pythagoreans. The Pythagoreans were trying to solve the geometric mean $a:b = b:c$, where a = 2 and c = 1, which led to the attempt to solve $b^2 = 2$ and thus led them to the discovery of the irrational number $\sqrt{2}$ (Struik 42). The idea of irrational numbers was first met with hesitation and some degree of fear, as it challenged the Greek's theory that all quantities could be represented by a ratio of whole numbers and forced them to delve deeply into the issue of continuity and the notion of the infinitesimal. Some people even claim that the Greek mathematician who discovered irrational numbers was drowned by fellow mathematicians as punishment for his heresy ("Lecture on Irrational Numbers"). The Greeks grappled with the concept of irrational numbers for some time before very hesitantly accepting them into their worldview. When Euclid published his tenth book of *Elements*, he included Theaetetus's theory of irrationals as a purely geometric concept (Struik 44). One reason why the concept of irrational numbers was so difficult for the Greeks was that it brought forward another issue facing mathematicians of the time, which was the notion of the infinite. At the time of the Pythagoreans' discovery of irrational numbers, the Greeks were struggling with the concept of the infinitesimal due to a set of paradoxes put forward by Zeno of Elea (Struik 42). This is the first record we have of any civilization even thinking about, let alone seriously discussing, a concept of infinity (Allen). The concept of the infinitesimal and its relation to the notion of irrational numbers was most fully fledged out by Dedekind and Weierstrass in the 19^{th} century and led to a rigorous construction of the set of real numbers (Warren). The inclusion of real numbers into our worldview and the acceptance of the infinitesimal that comes along with this acceptance has been extremely beneficial to both mathematicians and humanity at large. For example, in both physics and mathematics there are many famous irrational constants that are very useful.

While the Greeks were tackling the issues of irrational numbers and the infinite, they still did not consider negative integers to be valid numbers. This led them to consider many problems, such as $4x+20=0$, to be either absurd or unsolvable (Smith). The acceptance of negative integers into the definition of a valid number was a mental hurdle for the Greeks and many others; most likely because it is a very abstract notion that does not necessarily follow directly from the concrete act of measuring physical quantities. However, while Diophantus was rejecting negative solutions to equations, other civilizations like the Hindus had started to accept these solutions as possibilities, although skeptically (Struik 66). Liu Hui, a Chinese mathematician from the second century AD, was one of the first mathematicians to accept negative numbers into their worldview, going so far as to publish rules for how to add and subtract these types of numbers (Peng-Yoke). However, most mathematicians of the time still considered negative numbers to be invalid and absurd (Smith). The introduction of financial institutions and the concept of debt lent some validity to the concept of negative numbers and mathematicians slowly began to accept the idea. By the eighteenth century, many mathematicians; including Euler, Maclaurin, and Leibniz; had accepted negative numbers and were using them in their calculations with only some hesitation (Smith). This acceptance of negative numbers allowed for the solving of equations with negative roots and also proved useful for problems in physics and engineering.

The acceptance of negative numbers as valid was the first step towards the acceptance of imaginary numbers. While many people assume that the discovery of complex numbers came from the need to solve certain quadratic equations, such as $x^2+1=0$, the need for complex numbers actually became apparent from the attempt to solve cubic equations (Merino). Before the general acceptance of negative numbers as anything other than meaningless and absurd, it would have been an extra level of ridiculous to consider the square root of such a useless number. However, once negative was accepted into our definition of what a number is allowed to be, it paved the way forward for the beginnings of a discussion on the validity of what we now refer to as imaginary numbers. As previously mentioned, the desire to solve the general cubic equation was partially what forced the discovery and later acceptance of complex numbers. When dealing with the general cubic, it is always possible to rewrite the equation in a reduced form, as $x^3+px+q=0$, through a variable substitution ("Lecture on Cubic Equations"), which can in turn be split into the three separate cases where either q is negative, p is negative, or both p and q are negative. The case where both p and q are negative posed a problem for Cardano when he attempted to solve this case using the method of substituting in $x=u+v$ because the final solution would have a negative under a radical (Merino). In Cardano's book, *Ars Magna*, he omitted delving into this difficulty, but it clearly sparked some interest in the possibility of imaginary numbers, as he spent some time in chapter 37 performing calculations with $\sqrt{-15}$, albeit "tortuously" (Merino). Several other prominent mathematicians struggled with their feelings that these numbers were absurd while still recognizing their usefulness in calculations. Euler introduced the notation of i that we now recognize to represent $\sqrt{-1}$ and discovered the famous relationship between the exponential and trigonometric functions, $e^{ix} = \cos x + i \sin x$; which, when evaluated at $x = \pi$ gives the identity $e^{i} + 1 = 0$, which some refer to as “the most beautiful equation” (Coolman). Hamilton, Gauss, and Cauchy were three of the most influential mathematicians in the development of a rigorous definition of complex numbers (Merino). Hamilton discovered the set of quaternions, which is based off of the complex numbers and has been recognized as “the simplest associative number system of more than two units” (Struik 175). The inclusion of complex numbers into our idea of what a number could be has allowed for advancements in many fields, including (but not limited to) dynamics, electromagnetism, quantum mechanics.

The acceptance of complex numbers was the most recent time in history that our definition of what a number is allowed to be dramatically changed. As a student of mathematics, it is hard to imagine what it would be like to live in a world without imaginary numbers; let alone without irrational, negative, or rational numbers. And yet it was relatively recently that the definition of a number was narrower and less inclusive than what we are so used to today. Looking back, it’s easy to say that our ancestors were foolish or stupid for being so closed-minded, but really they were understandably hesitant to accept such drastic changes to their conception of numbers. It makes you wonder what mathematical discoveries lay ahead for us and how our definition of a number might change in the future.

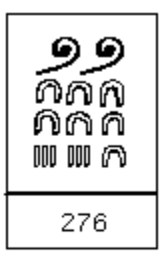
Table 1:



1	10	100	1000	10000	100000	10^6

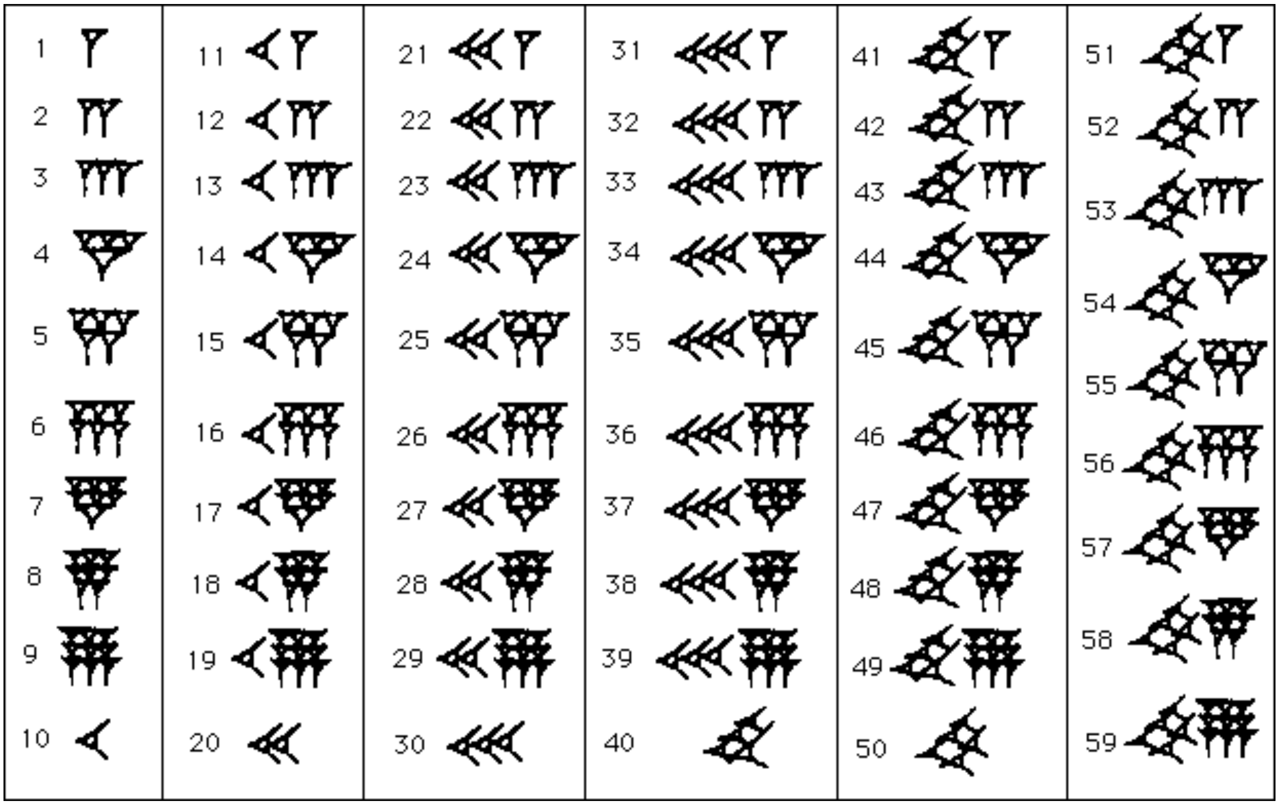
Egyptian numeral hieroglyphs

Table 2:



276

Table 3:



Babylonian numeral symbols

Table 4:

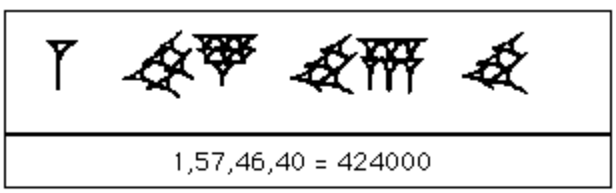
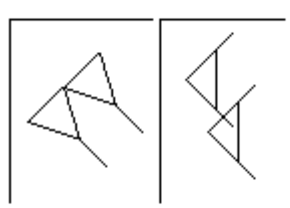


Table 5:



Babylonian Zeros

Works Cited

Allen, Donald. "The History of Infinity." Web. <http://www.math.tamu.edu/~dallen/masters/infinity/content2.htm>.

Coolman, Robert. "Euler's Identity: 'the most Beautiful Equation'." *Live Science* 2015. Web. <http://www.livescience.com/51399-eulers-identity.html>.

Geller, L. "Zero as a Placeholder." 1998. Web. <http://www.und.nodak.edu/instruct/lgeller/zeroph.html>.

Kelley, Loretta Anne. "Mathematical Devices, Early." *Encyclopedia.* 2002. Web. <http://www.encyclopedia.com/education/news-wires-white-papers-and-books/mathematical-devices-early>.

Lay Yong, Lam. "The Development of Hindu-Arabic and Traditional Chinese Arithmetic." *Chinese Science*.13 (1996): 33. Print.

Merino, Orlando. "A Short History of Complex Numbers." 2006. Web. <http://www.math.uri.edu/~merino/spring06/mth562/ShortHistoryComplexNumbers2006.pdf>.

Miller, Jeff. "Earliest Uses of Symbols for Fractions." 2014. Web. <http://jeff560.tripod.com/fractions.html>.

O'Connor, J., and E. Robertson. "Babylonian numerals." 2000. Web. <http://www-history.mcs.st-and.ac.uk/HistTopics/Babylonian_numerals.html>.

O'Connor, J., and E. Robertson. "Egyptian numerals." 2000. Web. <http://www-groups.dcs.st-and.ac.uk/history/HistTopics/Egyptian_numerals.html>.

Peng-Yoke, Ho. "Liu Hui." *Complete Dictionary of Scientific Biography.* Web. <http://www.encyclopedia.com/science/dictionaries-thesauruses-pictures-and-press-releases/liu-hui>.

Smith, Martha K. "History of Negative Numbers." 2001. Web. <https://www.ma.utexas.edu/users/mks/326K/Negnos.html>.

Struik, Dirk J. *A Concise History of Mathematics*. 4th ed. United States: Dover Publications, Inc., 1987. Print.

Wallin, Nils-Bertil. "The History of Zero." *YaleGlobal Online* 2002. Web. < http://yaleglobal.yale.edu/history-zero>.

Warren, Carryn Bellomo. "Irrational Numbers Outline." Web. <https://faculty.unlv.edu/bellomo/Math714/Notes/07_Irrational.pdf>.

Zeilberger, Doron. *Lecture on Cubic Equations*., March 2, 2017.

Zeilberger, Doron. *Lecture on Discovery of Irrational Numbers*., February 23, 2017.