Two Problems which Transformed Finance

Important Moments in the History of Mathematical Finance

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Abstract

In this paper, we discuss development of the concept of the *fair price* of a financial asset, i.e., the expected present value, by examining two historical problems in mathematics, the first "On a Soldier Receiving Three Hundred Bezants for his Fief" from the *Liber Abaci* of Leonardo of Pisa and the second the "Problem of Points" from Luca Pacioli's *Summa*. We consider each problem with its solution in their historical context, and then review a modern-day application of their results.

Introduction

Imagine a certain investor Mr. Smith wishes to purchase a financial instrument from ABC Finance. After T years, this asset will be worth on of several (possibly infinitely many) values. What price should Mr. Smith pay today for this asset?

The concept of *fair price* is essential to the operation of financial markets and several complex stochastic models, such as the Black-Scholes Derivative Pricing Model, exist to assist financial analysts evaluate the fair price of financial instruments, such as equity derivatives, corporate bonds, and even insurance products. Almost all these models, in one way or another, assess the fair price by calculating the *expected present value* (EPV) under a given set of assumptions. While an investor's, or in the case of insurance a policyholder's, willingness to pay will vary by personal preferences regarding risk exposure, duration of cashflows, and financial sector, the EPV gives an objective measure what *ought* to be paid for the procuration of said asset.

Use of the EPV is widespread in finance. In principle, the fair price of an asset today is the average value which the asset will take in the future, discounted to the present at some given interest rate. This principle relies on two components:

1. Implicitly, it relies on the *present value criterion* (PVC) or *net present value* (NPV), and

2. Explicitly, it relies on the mathematical expectation of a future value.

The remainder of this paper is dedicated to examining the historic, mathematical origins of these two components. As with most mathematics, a complete intellectual grasp of their origin requires a perusal of Greek and Ancient Hindu science and mathematics, however, for the sake of brevity, we must start regrettably late in their development in Medieval Europe, though the curious reader can find the rather sophisticated foundations of the European mathematics in the *Aryabhatiya* and the *Lilavati.*[1]

The origins of these two components are strikingly similar. Both can be traced to historically important word problems in mathematical instructional texts in Medieval Italy, although several centuries apart. The concepts themselves appear in the particular solutions to these two word problems. Notably, however, the solution introducing the NPV appears in the text in which the problem is initially proposed. The solution containing the second component does not appear until the seventeenth century. We speak of the "On a Soldier Receiving Three Hundred Bezants for his Fief" and the so-called "Problem of Points," contained in Fibonacci's *Liber Abaci* (Book of Calculations) and Luca Pacioli's *Summa de arithmetica, geometria, proportioni, et proportionalita* (*tr.* Summary of arithmetic, geometry, proportions, and proportionality), respectively.

1 Fibonacci and the PVC

The aforementioned NPV is the present value of cash inflows less the present value of cash outflows, i.e.,

$$NPV = \sum_{k=0}^{\infty} \frac{CF_{in,k}}{(1+i)^k} - \sum_{k=0}^{\infty} \frac{CF_{out,k}}{(1+i)^k}$$
$$= \sum_{k=0}^{\infty} \frac{CF_{in,k} - CF_{out,k}}{(1+i)^k}$$

And the PVC states that a company ought to undertake a project if and only if NPV > 0, assuming there is no uncertainty regarding the timing and amount of cash flows. A 2001 study published in the *Journal of Financial Economics* found that a majority of corporations rely on the PVC when making financial decisions, and concluded that the PVC is becoming increasingly prominent in the financial world.[2] In practice, the PVC allows entities to purchase and trade fixed-income securities, such as bonds, CDs, and low-risk mortgages.

Its importance established, we consider the first recorded use of the PVC in history in its context, formulation, and impact.

[1] Aryabhata *Aryabhatiya*, trans. Walter Eugene Clark. (Chicago: Chicago University Press, 1930), 38

[2] John R. Graham, Campbell R. Harvey, "The theory and practice of corporate finance: evidence from the field," *Journal of Financial Economics* 60 (2001): 197

1.1 The Context of the *Liber Abaci*

Twelfth and thirteenth century Europe was fragmented into city-states, territories, nations, and provinces, many of which issued their own currency. Growing trade within Europe and the emergence of prominent mercantile cities, which expanded existing trade with the Arab world and the Far East, necessitated the development of arithmetic tools for converting currencies and exchanging goods in a increasingly complex financial world. These trade networks economically connected Europe despite its patchwork of political borders. Calculations at this time were performed using abaci and the results recorded using Greek or Roman numerals. This method of computation was tedious, and the growing commercial world required a more efficient method.

Leonardo of Pisa, a commercial trade center in central Italy, supplied this need with the publication of his greatest work, the *Liber Abaci* in 1202. As a youth, Leonardo, alias Fibonacci, joined his father in Bugia in modern-day Algeria, where his father was employed as a customshouse official. From the brief biographical note in hist preface to the *Liber Abaci*, which supplies most of our knowledge of Fibonacci's life, we learn that according to his father's wishes, he studied mathematics and was introduced in Bugia to the "art of the nine Indian figures," i.e., 1 2 3 4 5 6 7 8 9. These figures, coupled with the Arab *zephir* (0), inspired Fibonacci to further pursue the study of mathematics in Bugia and abroad. In his own words:

> ...the introduction and knowledge of the art pleased me so much above all else, and I learnt from them, whoever was learned in it, from nearby Egypt, Syria, Greece, Sicily and Provence, and their various methods, to which locations of business I travelled considerably afterwards for much study...[3]

The *Liber Abaci* is the result of Fibonacci's studies; in it he compiled the mathematical methods he learned during his travels. The content of the book progresses from an introduction to Hindu-Arabic numerals through an audit of arithmetic operations and finally into algebra and geometry, presented through increasingly complicated word problems. Since the *Liber Abaci* predates Viètes *In artem analyticam isagoge* (1591), and the representation of numbers by letters, it contains very few equations. Its theorems and results are presented in *prose* form, so that reading through the text can grow rather tedious. Nevertheless, its structure indicates that it was intended as an instructional in commercial mathematics, and likely was used by Fibonacci to teach mercantile computation.

This mathematical textbook transformed Europe. Although they had previously appeared in part in Spain–notably in the *Codex Vigilanus* and Gerbet of Aurillac's *De multiplicatione et divisione*–and had been promoted by Pope Sylvester II in 999 A.D.,

[3]Leonardo, Fibonacci, *Liber Abaci* trans. Laurence Sigler (New York: Springer-Verlag, 2002), 3-4

the Hindu-Arabic numerals and the decimal system came to prominence in Europe in the wake of the *Liber Abaci.* The arithmetic algorithms it likewise introduced to Western Europe opened the floodgates for rapid financial development. The PVC, and other methods introduced by Fibonacci, enabled merchants and businessmen to issue and trade *lettres de faire*, which functioned as Medieval forward contracts. [4]. As historian William Goetzmann described in his most excellent essay *Fibonacci and the Financial Revolution,* "The mathematical capicity to calculate the present value of differing instruments led to a sustained period of financial innovation in a relatively localized geographical area."[5] Government bonds, corporate loans, and annuities could be now be priced, and thus came into existence in the 500 years following Fibonacci.

As said above, the PVC appeared in the solution to the problem "On a soldier receiving three hundred bezants for his fief," which appears in the Chapter Twelve of the *Liber Abaci,* and which we consider now.

1.2 The PVC's First Use: A Soldier's Fief

The premise is as follows:

> A certain soldier because of his fief received from a certain king 300 bezants each year, and it is satisfied in IIII payments, and in each payment he takes 75 bezants; this is a payment for three months by which necessity is collected together; he asks for a certain compensation in order to accommodate himself for interest because he accepts the 300 bezants of each payment, namely from payment to payment, of the capital and profit. Voluntarily acquiescing to this he invests the bezants at a profit of two bezants per hundred in each month. It is sought how many bezants he makes in his investment.[6]

In brief: a soldier receives 75 bezants quarterly, which he invests at 6% quarterly. The annuity is changed to a year-end payment of 300 bezants. What is the effect on the soldiers income?

To answer this question, Fibonacci applies a method he calls the "method of trips," introduced earlier in the text. This involves finding the present value of the annuity by multiplying the 75 bezants paid quarterly by the geometric series of the discount factors, which in Fibonacci's notation comes to $\frac{50}{53}\,\frac{50}{53}\,\frac{50}{53}\,\frac{50}{53}$. This series is computed as (in our notation):

$$\frac{50 \times 53^3 + 50^2 \times 53^2 + 50^3 \times 53 + 50^4}{53^4} = \frac{50}{53} + \left(\frac{50}{53}\right)^2 + \left(\frac{50}{53}\right)^3 + \left(\frac{50}{53}\right)^4$$

[4]"The Origins of Derivative Instruments" *McGraw Hill Education*, accessed April 24, 2017 http://highered.mheducation.com

[5]William Goetzmann. "Fibonacci and the Financial Revolution," *National Bureau of Economic Research Working Paper Series* 10352 (2004): 8

[6]Sigler, 392

Thus the present value of the annuity is found to be, again in Fibonacci's notation, $\frac{33}{53}\frac{6}{53}\frac{42}{53}\frac{46}{53}259$ bezants. This is approximately 259.882921 bezants. A modern mathematician or economist would likely have calculated the the present value using the quantity 75 $a_{\overline{4}|,0.06}$, where

$$a_{\overline{n}|,i} = \frac{1-(1+i)^{-n}}{i}$$

It is left to the reader to verify that these solutions are equivalent.

It is left unanswered what compensation the soldier receives from the king to "accommodate himself for interest," likely because Fibonacci has previously demonstrated the mechanism by which the 300 bezant year-end payment can be discounted to the present, and it remains only to subtract this from the above answer. The present value of the 300 bezants is approximately 237.63 bezants, and thus the soldier ought to be compensated 259.88-237.63 = 22.25 bezants for interest lost on his investment.

Fibonacci's explicit statement that the soldier is entitled to some compensation, even though the compensation is only implicitly given, showcases the principle now underpinning all financial analysis, namely, that portfolios–even those with differently-timed cashflows–can be quantitatively compared using the present value.

Returning briefly to our discussion of the fair price of a financial asset at the beginning of this essay, we note that oftentimes the fair price of a fixed-income asset is oftentimes the present value of the cashflows. More generally, the fair price of a fixed-income is determined by setting the present value of payments equal to the present value of the income to be received, i.e., the NPV of both companies is 0. For example, if Mr. Smith is to make 10 annual payments of $\$P$ to ABC Finance in return for 30 annual payments of \$100, starting 11 years from now P is determined such that $\$P\ a_{\overline{10}|,i} - (1+i)^{-10} a_{\overline{30}|,i} = 0$, or

$$P = (1+i)^{-10}\frac{a_{\overline{30,i}|}}{a_{\overline{10,i}|}}$$

For both Mr. Smith and ABC Finance, the NPV of their cash in/outflows is 0.

2 The EPV: Problem of Points

As previously mentioned, the *Liber Abaci* introduced the Hindu-Arabic numbers to Western Europe. Their use, however, was slow to catch on, and didn't come to widespread use until the fourteenth century, more than 100 years after the book's publication.[7] In the centuries following Fibonacci, practical mathematics, including the mathematics of commerce, became the practice of the so-called "reckon masters" rather than the Academics. Very little mathematical advancement was made during this time beyond that which was already made by the Greeks and Arabs.

[7]Dirk J. Struik, *A Concise History of Mathematics*(London: G. Bell and Sons LTD, 1954), 105

2.1 The Formulation of the Problem: Luca Pacioli

In late fifteenth century Italy, a Franciscan Friar named Luca Pacioli compiled all arithmetic, algebra, and geometry known at that time into a single-volume work known as the *Summa de arithmetica, geometria, proportioni, et proportionalita* (*tr.* Summary of arithmetic, geometry, proportions, and proportionality). Published in 1494, it became one of the first mathematical books to be printed, and became a standard reference not only for mathematics, but also for accounting, since it contains the first published instruction on double-entry bookkeeping. It also contains the first published description of the problem of points, which it discusses in several problems, the first of which we reproduce below. In short, the problem of points considers how to divide a prize pot between two players, who, having agreed to reward the entire pot to the first player to reach a certain number of points, must end the game prematurely and divide the pot between them. As Pacioli writes:

> A company plays a ballgame to 60 and each goal is 10.
> They stake 10 ducats in all. It happens by certain accidents they are not able to finish; and one party has 50 and the other 20. One asks what portion of the stake is due to each party. For this problem I have found different opinions, going in the one direction to the other, all appear to me incoherent in their arguments. But the truth is what I will say, together with the right way.[8]

Pacioli then presents three equivalent solutions to the problem, all of which distribute the 10 ducats to the two companies based on the score of the game when the game prematurely ended, that is, *he only considered the past.* For the sake of brevity, we only consider the first.

> First, consider how many goals at most are able to be made between the one and the other party; this will be 11, that is, when they are at 50 points each. Now you see what this part with 50 has of all these goals; this gives $\frac{5}{11}$; and 20 gives $\frac{2}{11}$. Therefore, from this the one party must take $\frac{5}{11}$ parts and the other party $\frac{2}{11}$ parts. The sum makes $\frac{7}{11}$. Then $\frac{7}{11}$ is worth 10. What is due with $\frac{5}{11}$ and what with $\frac{2}{11}$? Thus to the one with 50 will come $7\frac{1}{7}$ and $2\frac{6}{7}$ to 20. Finished.[9]

This solution, although straightforward, presents rather counter-intuitive results if the game is stopped after 10 points are rewarded, i.e., only 1 round has been played. In this case and by this method, one company takes all ducats, for to mirror Pacioli's reasoning:

[8] Richard J. Pulskamp. "Summa de Arithmetica Geometria Proportioni Et Proportionalita," *Xavier University*, 2009, accessed April 24, 2017 https://cerebro.xu.edu/math/Sources/Pacioli/summa.pdf

[9] Pulskamp

Noting that the maximum number of goals which can be made between the two parties is still 11, the one party with 10 points has a part $\frac{1}{11}$ of all these goals, and the other party has 0. The sum makes $\frac{1}{11}$, which is worth 10 ducats. Then the party with $\frac{1}{11}$ is due 10 ducats and the other 0 ducats.

As any gambler is aware, the first-round winner is not always the inner of the entire game. It seems fundamentally unfair now to award all 10 ducats to the winner of one round. A good solution to the problem of points ought to offer a each player a proportion of the pot according to the points they have earned, but must avoid this paradox.

Many attempted solutions to the problem of points appeared from 1495 to 1694, notably by Cardano, Tartiaglo, and Forestani. None of these solutions, however, appear correct, since they counted only past rounds and disregarded future potential outcomes. Furthermore, although not a historical criticism, these solutions lead to more counterintuitive results when the players are unequally skilled, that is, when one player has a higher probability of winning each round than the other.

2.2 The Probabilistic Solution: Fermat and Pascal

In 1654, a French aristocrat, writer, and gambler the Chevalier de Méré posed the problem of points, although in a different form than was initially posed by Pacioli, to the two great French mathematicians Blaise Pascal and Pierre de Fermat. The two solved the problem in correspondence via letter. The version of the problem posed to them differs from Pacioli in that the two players play to three points for a pot of 64 pistoles, to which each have contributed 32 pistoles. Although the point mechanism is not explicitly stated, Pascal and Fermat imply in their letters that as each round is played, both players have equal chances of scoring the point.

Fermat proposed a correct, but rather inefficient solution. In sum, Fermat solves the problem of points by writing out all possible future outcomes from the point at which the game halted, and then computing the chances of each player winning. The pot should be awarded according to each player's chance.

To illustrate this method, assume that the game is stopped in the third round. A maximum of five rounds can be played. Assume player A has won 2 points and player B has won 1, i.e., score is 2-1. Finishing the game requires a maximum of two rounds to be played. Thus, we have four possible outcomes (note: For ease of computation, Fermat assumes that the players can play "pointlessly," and may continue even after the game has been won):

1st Round	Score	2ndRound	Fin. Score	Winner
A	3-1	A	4-1	A
A	3-1	B	3-2	A
B	2-2	A	3-2	A
B	2-2	B	2-3	B

As seen in the table, A wins in $\frac{3}{4}$ of the scenarios. Thus A should receive $64 \times \frac{3}{4} = 48$ pistoles and B should receive 16 pistoles.

Pascal acknowledged the logical soundness of this method, but also noticed its inefficiency. In his July 29, 1654 letter to Fermat he presented an abridgment to this method, which we paraphrase for simplicity. Assume first that player A and player B both have two points. The next round, the fifth, will award the winner all 64 pistoles, with equal chance of either winning. Thus, if the players wish to end the game before playing the fifth round, each should take back his 32 pistoles.

Now consider the third round, assuming A has 2 points and B has 1 point. If A wins, then he can take home the 64 pistoles. If B wins, then the game is 2-2 and the two players are effectively entitled to 32 pistoles each, as seen above. Thus if the game is going to end before the fourth round, as is, then A has a 50-50 chance of winning 64 or 32 pistoles. Thus he should receive $\frac{1}{2} \times 64 + \frac{1}{2} \times 32 = 48$ pistoles.The entire scenario can be plotted out recursively using a binomial tree. Denote the amount of prize money A should receive in pistoles assuming the game ends after each round is played by p_A and the number of rounds A has won by n_A. Then $p_B = 64 - p_A$ and $n_A =$ number of rounds played $- n_A$.

This table is derived as above by considering each end-of-game scenario, identifiable in round j by $n_X = 3,\ p_X = 64$, and then deriving for round $j - 1$ the value p_X as the expected value of the two reachable outcomes for round j.Note that the table omits cases such as $n_A = 4, n_B = 1$ since these cases can only occur after the game has ended.

Table 1: Pascal's Method for Solving the Problem of Points

Rd. 1	Rd. 2	Rd. 3	Rd. 4	Rd. 5
		$n_A = 3,\ p_A = 64$ $n_B = 0,\ p_B = 0$		
	$n_A = 2,\ p_A = 56$ $n_B = 0,\ p_B = 8$		$n_A = 3, p_A = 64$ $n_B = 1, p_B = 0$	
$n_A = 1,\ p_A = 44$ $n_B = 0,\ p_B = 20$		$n_A = 2,\ p_A = 48$ $n_B = 1,\ p_B = 16$		$n_A = 3,\ p_A = 64$ $n_B = 2,\ p_B = 0$
	$n_A = 1,\ p_A = 32$ $n_B = 1,\ p_B = 32$		$n_A = 2,\ p_A = 32$ $n_B = 2,\ p_B = 32$	
$n_A = 0,\ p_A = 20$ $n_B = 1,\ p_B = 44$		$n_A = 1,\ p_A = 16$ $n_B = 2,\ p_B = 48$		$n_A = 2, p_A = 0$ $n_B = 3, p_B = 64$
	$n_A = 0,\ p_A = 8$ $n_B = 2,\ p_B = 56$		$n_A = 1,\ p_A = 0$ $n_B = 3,\ p_B = 64$	
		$n_A = 0,\ p_A = 0$ $n_B = 3,\ p_B = 64$		

Pascal's method of solution avoids the unnecessary computations seen in Fermat's method. Furthermore, his method is easily generalized to the case when the probability that A wins a round is greater than the probability that B wins. This is not true of Fermat's method, although this criticism was not made in the letters between the two. These two methods also avoid the counterintuitive result of Pacioli's method. Indeed, when only one round is played, the winner of the round is awarded a higher proportion of the pot than the loser, but does not take all.

By dividing the pot based on the expected value of the future winnings of each player, Pascal and Fermat supplied the tools needed to price financial assets *under uncertainty.* To again briefly return to our introduction, when Mr. Smith purchases a financial asset that carries a degree of risk, as said above, he would have to pay the EPV of the asset. From any given point in time, this price can be found by discounting all possible cashflows at the given (or even variable) rate of interest and taking the probability-weighted average. Even today, financial assets are being priced (via computer) with models based off the reasoning used by these two seventeenth century French mathematicians to solve a fifteenth century Italian problem.

3 Pricing Models: Where Fibonacci and Pascal Collide

To complete this paper, we consider a widely used asset pricing model. The financially literate reader may have recognized Pascal's method as being very similar to the *Binomial Options Pricing Model* (BOPM) proposed by Cox, Ross, and Rubinstein in 1979. Indeed, the Cox et. al applied almost identical reasoning to the pricing of financial options as Pascal did to the division of stakes. The illustrate this method, and demonstrate the utility of Fibonacci and Pascal's techniques even in the modern era, we consider the following question:

Question: Assume that Mr. Smith owns a stock currently valued at \$100 and wishes to hedge his risk over the next year. The stock's price moves in discrete ticks, either up or down by 20%. The first tick will occur in 6 months time, and the second in one year's time. At each movement, the probability of ticking upward is $\frac{4}{7}$, and the semi-annual interest rate is 5%

1. Assume Mr. Smith wishes to purchase a European put option on this stock with a strike price of \$100, expiring one year from now. This will allow him, regardless of the price of the stock at year's end, to sell the stock at \$100. What is the fair price of this option?

2. Immediately after the purchase of this put, Mr. Smith wishes to upgrade to an Bermudan put option, that will also allow him to sell the stock, should he choose, six months from now. How much should Mr. Smith pay for this upgrade?

Solution: The solution to 1. can be found using Pascal's method. Assume that in six months time, the stock ticks upward. Then, if at year's end it ticks upwards again, the put option is worthless, since the stock is priced at \$144 and Mr. Smith will not choose to sell it at \$100. But if it ticks downward at year's end, its valued is \$96 and Mr. Smith will exercise his put option for a payoff of \$4. Thus, given an upward tick, the put option's price in 6 months is $\left(\frac{4}{7} \times \$0 + \frac{3}{7} \times \$4\right) \div 1.05 \approx \$1.63$. By similar reasoning, its value in six months given a downward tick is approximately \$16.87. Thus its fair price today is $\left(\frac{4}{7} \times \$1.63 + \frac{3}{7} \times \$16.87\right) \div 1.05 \approx \$7.77$ Note that if Mr. Smith wished to sell the option at the end of six months, the price he could sell it at would be its value at that time, either \$1.63 or \$16.87.

The solution to 2. requires us to use a variant of Fibonacci's method of solution. To solve, we must find the EPVs of the pre-and-post-upgrade cashflows. The difference will be the price of the upgrade. We've already calculated the EPV pre-upgrade. With the upgrade, the year's-end cashflows are identical, so there is no need to revisit them. Given an upward tick, the non-upgraded option is worth \$1.63. If Mr. Smith were to chose to exercise the option, he would sell the stock for \$100 when it is priced at \$120. Therefore he will not exercise, and the value of the upgraded option at this point is still \$1.63. Given a downward tick, however, exercise of the upgraded option would make Mr. Smith \$20 (he sells a \$80 for \$100), which is higher than the EPV of the option if he chooses not to exercise it (\$16.87). Thus Mr. Smith would exercise the upgraded option in six months time if the stock were to tick downwards. Thus, the EPV of the upgraded option is $\left(\frac{4}{7} \times \$1.63 + \frac{3}{7} \times \$20\right) \div 1.05 \approx \$9.05$. Thus Mr. Smith would need to pay \$9.05 - \$7.77 = \$1.28 for the upgrade.

Conclusion

The development of financial and financially useful mathematics did not stop with these two problems. The discovery of Brownian motion, the formulation of Ito's Lemma, and the publication of the Black-Scholes framework define three more transformative events in the history of mathematical finance. Nevertheless, financial mathematicians and analysts today owe a debt of gratitude to Fibonacci, Fermat, and Pascal. The widespread use of their methods even centuries later attests to the great impact of their work on the course of financial history. Without the results of their work, the stocks that raise capital for emerging companies, forwards and futures that protect farmers and producers from potentially devastating price drops, and the securities that make up retirement portfolios could not be adequately priced, and would likely not even exist. It is for this reason that we can call their origins, *On a soldier receiving 300 bezants for his fief* and *the problem of points* two problems which transformed finance.

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