Leonardo Pisano Fibonacci

By Susmita Paruchuri

Born in 1170, Leonardo Pisano was a medieval Italian mathematician most famous for his been *Liber Abaci* (1202), or “Book of the Abacus” (Gies). He is known as “Fibonacci,” coming from “filius Bonacci,” which translates to “son of Bonacci” in Latin. Bonacci was not his father’s name, so the name can better be translated as “of the Bonacci family.” He is also sometimes known as “Bigollo,” which means “traveler” in Tuscan dialect, as his travels were a large part of how he was able to write such revolutionary works. His father was named Guilichmus (translating to “William”) and he had a brother named Bonaccinghus — there is very little written on him, and we do not know anything about his mother, or whether he was ever married or had children (Devlin).

There is little known about his life beyond his mathematical writings. Fibonacci was born in Italy and educated in North Africa, because his father held a diplomatic post there representing merchants of the Republic of Pisa, trading in Bugia, now called Begaia, which is a Mediterranean port in northeastern Algeria. Fibonacci learned mathematics there and travelled with his father.

By 1200, Fibonacci wrapped up his travels and returned to Pisa and began to write a series of texts. Of these, the titles we have copies of include *Liber abbaci* (1202), *Practica geometriae* (1220), *Flos* (1225), and *Liber quadratorum*. Because Fibonacci lived before printing was possible, all his books were handwritten, and so having even the works we have is a rarity. Still, there are other works that are known to have existed and are now lost, such as *Di minor guisa*, or “book in a smaller manner,” a book on commercial arithmetic aimed at merchants, that historians believe was responsible for popularizing new methods in European mathematics (Devlin). His commentary on Book X of Euclid's *Elements* is also nowhere to be found, on a numerical treatments of irrational numbers, over Euclid's geometric treatment (School of Mathematics and Statistics).

Liber abbaci was Fibonacci's first major work and serves as a review of basic arithmetic and algebra. While the first edition has not survived, Fibonacci's self-edited second edition in 1228 still exists. In it, he writes about common finger computations and Roman numerals, and then introduces Indian numerals. This book introduced the "Hindu-Arabic" numerical system and algebraic solution techniques for different types of problems, knowledge he acquired from Arab sources while he was abroad — he observed Muslim traders using the numbers 0 to 9 and manipulating them through certain rules rather than finger-counting or a physical counting table (Devlin). He drew from the works of al-Khwarizmi, Abu Kamil and al-Karaji, covering a variety of applied problems such as cost and profit, barter, partnership, investment, alligation, mensuration, and simple geometry. *Liber abbaci* was a turning point in western mathematics, establishing a style for commercial arithmetic books in Europe, and included the famous "Fibonacci sequence" (Swetz). The publication of this book made him famous throughout Italy, and was even summonsed to an audience by the Emperor Frederick II (Devlin 15)

Of his other books that have made it to the 21st century, *Practica geometriae* discusses geometry based on Euclid's texts. Fibonacci teaches the reader how to find square and cube roots, work with tables for indirect measurement, find the dimensions of linear and curved surfaces and solids, compute using Pisan units, and analyze polygons — it was a book intended for medieval landmeasurers, or surveyors (Hughes).

This book is dedicated to Dominicus Hispanus and includes theorems based on Euclid's famous books, *Elements* and *On Divisions*. It also includes practical information for the professions it was intended for, such as the chapter dedicated to calculating the height of tall objects by using similar triangles.

Flos ("The Flower") discusses the problems presented to Fibonacci in a public display by another mathematician in the Court of the Emperor Frederick II and his solution. One of these problems is to solve $x^3 + 2x^2 + 10x = 20$, which Johannes of Palermo challenged him to solve, taken from the algebra book by Omar Khayyam, in which it is solved using the intersection of a circle and hyperbola (School of Mathematics and Statistics). In *Flos*, Fibonacci comes up with his own solution via an original method, saying Euclid's method of solving equations using square roots would not work in this problem. He gives his answer in sexagesimal notation, with an approximation more accurate than the solutions of any Arab mathematician of the time. He does not explain his methods, he simple gives the answer 1.22.7.42.33.4.40, translating to the base 10 decimal 1.3688081075. The next problem was:

> "Three men owned some money, their shares being 1/2, 1/3, and 1/6. Each took some money at random until none was left. The first man then returned 1/2 of what he had taken, the second 1/3 and the third 1/6. When the money now in the pile was divided equally among the men, each possessed what he was entitled to. How much money was in the original store, and how much did each man take?" (Horadum)

Lastly, *Liber quadratorum*, or "book of squares," published in 1225, is a text on number theory that includes methods used to find Pythagorean triples (Devlin). Fibonacci solves Diophantine problems involving second degree equations with two or more variables, with solutions required to be given as integers or rational numbers.

In *Liber quadratorum*, a statement now called Fibonacci's Identity appears. It states that the product of two sums of two squares is itself, also a sum of two squares. To illustrate, $(1^2 + 4^2)(2^2 + 7^2) = 26^2 + 15^2 = 30^2 + 1^2$. This is also sometimes known as Brahmagupta's Identity in honor of the Indian mathematician who came before Fibonacci and had the same conclusions. Generalized, the identity reads $(a^2 + b^2)(c^2 + d^2) = x^2 + y^2$ for some x and y. Both Fibonacci and Brahmagupta showed that x can be written as $(ac + bd)$ and y can be written $(ad - bc)$, and using simple algebra, if we square each of these terms and add them together, we get $(a^2 + b^2)(c^2 + d^2)$.

While he was inspired from his travels abroad, there are many examples contained in his books that do not have equivalents in Arabic literature, although he does quote Al-Khwarizmi, for example, in his discussion of $x^2 + 10x = 39$. He also includes a seemingly original, "remarkably mature" proof that the roots of $x^3 + 2x^2 + 10x = 20$ cannot be expressed in terms of Euclidean irrationalities $\sqrt{a + \sqrt{b}}$, and so they cannot be constructed by just a ruler and compass. He proved this by checking Euclid's fifteen cases rigorously, solving the positive root of the equation and finding six sexagesimal places (Struik 104).

But aside from his books, Fibonacci is most famously known for introducing Europe to what is now called the "Fibonacci Sequence," or Fibonacci Numbers. He first discovered this famous sequence while studying how rabbits breed in a practical problem in *Liber abbaci.* After each month, he noted that the number of pairs of rabbits would increase from 1 to 2 to 3 to 5 to 8 to 13, and so on, with each term being the sum of the previous two terms — or $F_n = F_{n-1} + F_{n-2}$ (Medieval Mathematics).The rabbit problem assumes the following: we begin with one male and one female, rabbits reach sexual maturity after one month, a rabbit's gestation period is one month, after reaching sexual maturity, female rabbits give birth every month, and a female rabbit always gives birth to one male rabbit and one female rabbit.

The first pair cannot mate until after two months, because they have not reached sexual maturity, so for the first two months there is one pair of rabbits. After the the third month, the first pair gives birth to another pair — now we have two pairs. In the next month, the first pair gives birth again but the second pair is too young — now we have three pairs. This continues for a year, at which point we have 233 pairs (Hom).

This infinite sequence was actually already discovered by Indian mathematicians but many of the relationships associated with it were not discovered until centuries after Fibonacci died. The first known reference is in the *Chandashastra*, or "The Art of Prosody," by the Sanskrit grammarian Pingala between 450 and 200 BCE. The Indian mathematician Virahanka demonstrated how the sequence in used in the analysis of meters with long and short syllables in the sixth century. In the 1150s, Jain philosopher Hemachandra also wrote a text on this sequence (Devlin).

Firstly, the way the sequence regenerates itself was a new discovery by Fibonacci. Every third number is divisible by 2 ($F_3 = 2$), every fourth number is divisible by 3 ($F_4 = 3$), every fifth number is divisible by 5 ($F_5 = 5$), every sixth number is divisible by 8 ($F_6 = 8$), every seventh number is divisible by 13 ($F_7 = 13$), and so on.

Despite the rabbit problem's unrealistic nature, this sequence appears to be pervasive in nature. For example, there are many species of flowers with numbers of petals in the Fibonacci sequence, pineapples' spiral arrangements occur in patterns of 5s and 8s, pinecones' arrangements in 8s and 13s, and the sunflower seeds in 21s, 34s, 55s and even higher.

Fibonacci did not directly discover all these applications of the sequence; his name was attributed to it in 1877 when Eduouard Lucas made the decision to pay tribute to him. In the 1750s, Robert Simson made another discovery about the Fibonacci sequence — the ratio of each term to its previous term converges to approximately 1 : 1.6180339887, which is an irrational number that is equal to $\frac{1+\sqrt{5}}{2}$. This is called the Golden Ratio, or the Golden Mean, Golden Section, or Divine Proportion, and the Greek letter phi (ϕ). Two numbers are in the Golden Ratio if the ratio of the sum of the numbers to the larger number is equal to the ratio of the larger number to the smaller one.

The Golden Ratio has its own unique properties: for example, $1/\phi = \phi - 1$ (or approximately 0.618…), and $\phi^2 = \phi + 1 = 2.618...$. A rectangle with sides of the ratio $1 : \phi$ is called a Golden Rectangle, and many artists and architects in history have proportioned their work using Golden Rectangles and the Golden Ratio, especially Renaissance artists such as Leonardo da Vinci — it appears in the Mona Lisa — but even as far back as ancient Egypt and Greece. (Medieval Mathematics). Works such as the Parthenon in Athens have been constructed using the Golden Ratio. Also, ϕ is the ratio of a regular pentagon's side to its diagonal, and the pentagram that results forms a star, which is the same shape of the stars that are used in many countries' flags (Hom).

Also, an arc that connects opposite points of nested Golden Rectangles forms a logarithmic spiral, also known as a Golden Spiral — these are found in many instances in nature: shells, flowers, animal horns, human bodies, storm systems, and even in entire galaxies.

Again, while the Fibonacci sequence was not a major element of *Liber abbaci* and he was not the person who discovered any of the notable properties about the sequence, the book still discusses other famous mathematical problems, including the Chinese Remainder Theorem, prime numbers, perfect numbers, formulas for arithmetic series and square pyramidal numbers, Euclidean geometric proofs, and also studied simultaneous linear equations similar to Diophantus an Al-Karaji. In this book, he also introduced lattice multiplication, a way to multiply large numbers using a grid with diagonal lines and splitting the numbers to be multiplied into their place values. This method was originally brought up by Islamic mathematicians such as Al-Khwarizmi (Medieval Mathematics).

Liber abbaci is also one of the ways the Hindu-Arabic number system was introduced into Western Europe. They were imported centuries before Fibonacci by merchants, scholars, ambassadors, pilgrims, and soldiers from Spain and the Levant. These numerals were met with opposition from the public, and it was even forbidden for bankers in Florence to use them, but during the 14th century Italian merchants began using them for bookkeeping (Struik 105).

Again, since we do not know much about Fibonacci aside from his mathematical writings, the date of his death is not known, but it was most likely in Pisa between 1240 and 1250. While his name is broadly known, his influence was actually quite limited — he made many mathematical contributions, but he is mostly linked to his role spreading the use of Hindu-Arabic numerals upon his return from his travels and his rabbit problem in *Liber Abbaci*.

However, his work in number theory was very much overlooked during the Middle Ages — the same results appear in Maurolico's work 300 years later — and his work in practical problems made him a teacher of computation and of surveyors. He also taught the "Cossists," who took their name from the word "causa," instead of "res" or "radix," first used in Europe by Fibonacci. Since we know so little about Fibonacci, these contributions are likely to only scrape the surface of all his accomplishments (School of Mathematics and Statistics).

Works Cited

Devlin, Keith J. *Finding Fibonacci: the quest to rediscover the forgotten mathematical genius who changed the world*. Princeton, NJ: Princeton U Press, 2017. Print.

Gies, Frances Carney. "Leonardo Pisano." *Encyclopædia Britannica*. Encyclopædia Britannica, inc., 20 July 1998. Web. 01 May 2017.

Hom, Elaine J. "What is the Fibonacci Sequence?" *LiveScience*. Purch, 14 June 2013. Web. 01 May 2017.

Horadum, A.F. "Eight Hundred Years Young." N.p., n.d. Web. 01 May 2017.

Hughes, Barnabas. "Fibonacci's De Practica Geometrie." *Barnabas Hughes | Springer*. Springer-Verlag New York, n.d. Web. 01 May 2017.

"Leonardo Pisano Fibonacci." *Fibonacci biography*. School of Mathematics and Statistics, n.d. Web. 01 May 2017.

"Medieval Mathematics." *The Story of Mathematics*. N.p., n.d. Web. 01 May 2017.

Swetz, Frank J. "Fibonacci's Liber Abaci." *Fibonacci's Liber Abaci | Mathematical Association of America*. N.p., n.d. Web. 01 May 2017.