Paradoxes in Mathematics World

The study of paradoxes started from the early time of ancient Greece and it is still one of the most interesting topics in the mathematical world. Paradoxes are contradictions of understanding, contradictions of logic, contradictions of semantic and contradictions of thinking. Paradoxes may be disturbing; their study may reveal inadequacies, confusion or incoherence in some of our most deeply entrenched principles and beliefs. But they can be also instructive as well as fun (Olin, pg1). Many mathematicians like Zeno and Russell devoted their lifes to the study of paradoxes and many mathematicians participated in the study process and during the process they get a more comprehensive understanding of paradox through the definition, origin and solution.

The definition of paradox is that "The term *paradox* comes from the ancient Greek terms for *against* or *beyond* (*para παρά)* and *expectation* or *opinion* (*doxa δόξα*). "The Greek terms emphasize the counterintuitive nature of paradoxes. Our intuitions about the world, then, are central to what it means to be a paradox" (Cuonzo, pg18). Broadly construed, "a paradox can be anything from a tough problem or a counterintuitive opinion or conclusion to a visual sleight of hand" (Cuonzo pg2). Paradoxes are not only in math, and it is in such a diverse area like economics, biology and thermodynamics.

There are three main ways of defining paradox, namely as (1) a set of inconsistent statements, in which each statement seems true (Rescher, 2001), (2) an argument with seemingly good assumptions, seemingly fine reasoning, but an obviously false conclusion (Mackie, 1973), (3) an unacceptable conclusion derived from seemingly good premises using seemingly good reasoning (Sainsbury, 2009). The most common paradox in life might be the seemingly unacceptable conclusions derived by seemingly acceptable reasons. "The frequent words *seems* and *seeming* suggest that paradoxes involve conflicts among seemingly unproblematic elements" (Cuonzo, pg17). And "in other words, they force us to question whether our intuitive understanding of the world is really accurate" (Cuonzo, pg18).

Paradoxes show people absurdities with a seemingly perfect reason. People are confused about this because the results change and challenge the credibility of reason. People live their lives with logic and use logic to predict and behave daily, but how can people still believe in logic if the contradiction and false conclusion is where the logic can lead.

In order to solve the paradox problems, people should have a clear understanding that which part indeed cause the consequence so they can just focus on how to fix that part.

To resolve a paradox with seemingly unacceptable conclusion, we should show that the paradoxical argument does not in fact present us with a flawless operation of reason. Therefore, generally, we have two options in offering a resolution for a paradox: (A) we might eliminate the illusion that the argument is rigorous by pointing out a fallacy in the argument; or (B) we might explain away the appearance of fault in the conclusion. In other words, we need to explain why the conclusion seems false but it is actually true. To accomplish (A) – finding a problem in the argument, we could: (a) point out at least one of the premises has some nonobvious flaw; or (b) claim that the reasoning is invalid.

To resolve a paradox with inconsistent conclusions, there are two ways. One is to point out some conclusions are not true by looking in the arguments in the reasons. And the other way is to try to explain why these conclusions might be true by looking at the contradictions.

Famous mathematical paradoxes

1.Zeno's paradox

Zeno's first paradox is about Achilles and tortoise. In a race, the quickest runner can never catch up with the slowest runner, since the fastest runner must first pass the point where the slower one started to run, so that the slower must always hold a lead.

In the paradox of Achilles and the tortoise, Achilles is in a running game with the tortoise. Achilles gives the tortoise a head start of 100 meters. If we suppose that each racer runs at some constant speed (one very fast and the other very slow), then after they run for a time, Achilles will run 100 meters, He will be at where the tortoise started. During this time, the tortoise ran a shorter distance, say, 10 meters. It will then take Achilles some more time to run that 10m, by the time Achilles finished the 10m, the tortoise will have ran farther, and then more time still to reach this third point, while the tortoise moves ahead. Thus, whenever Achilles reaches somewhere the tortoise has been, he still has some more to go. Therefore, because there is an infinite number of points Achilles has to pass through where the tortoise has already been, he can never catch up with the tortoise.

Zeno's second paradox is the Dichitomy paradox. The question is that can you walk across a room? When you are walking, you must walk across half of the distance. Then you walk half of the distance of the remaining distance. And then you walk half of the distance of the new remaining distance. There would be infinitely many midpoints in the room so as a conclusion, no one can walk through this room without infinitely moves and infinitely time.

Zeno's third paradox is about the arrow paradox. "If everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless."(Wikipedia) In the arrow paradox (also known as the fletcher's paradox), Zeno says that for a motion to occur, an object must change the original position which it occupies. He uses arrow in flight as an example. He says that in any point of time, the arrow is neither moving to where it will go to, nor to where the arrow is not at. It cannot move to where it is not, because there will be no time elapses for it to move there; it cannot move to where it is, because it is already there. In other words, at every moment of time there is no motion happening. "If everything is motionless at every instant, and time is entirely composed of instants, then motion is impossible. Whereas the first two paradoxes divide space, this paradox starts by dividing time—and not into segments, but into points."(Wikipedia)

Zeno's paradoxes are believed to be solved by Georg Cantor in the nineteenth century with his self-invented method transfinite arithmetic and most philosophers and mathematicians believe in it too. Zeno actually set a speed limit for his paradox and in this way it is true that there is no way to pass through the whole room. But a person can run or walk much faster to pass across the whole room in which infinitely many actions can be done in a finite interval of time. Suppose a person can walk across a room faster and faster. He walks across half of the room in 6 seconds and then the following half distance in 6/2 seconds and then the following half in 6/4 seconds and so on and so forth. Then he will be able to walk across the whole room in 12 seconds.

The possibility of hypertasks has been questioned by a lot of people and J.F Thomas (1970) tried to prove that doing infinitely many actions is not possible by giving the following example. Suppose there is a lamp and there will only be one button on the lamp. The button will turn on the lamp if it is originally turned off or turn off the lamp if it is originally on. Because the lamp is originally at a turned-off situation, then it could be saying that if the lamp is pressed for an odd number of times, the lamp will be on and of course if the lamp was pressed in an even number of times the lamp will be off. Now suppose that he tried to pass the button for an infinite number of times by press the first time in one minute, and the second time in the next half minute, press the third time in the next quarter minute and so on. So the question is is the lamp on or off at the end of two minutes? Here the paradox happened. It cannot be on because Thomas never turned it on without turning if off. The other way works the same.

2. Eubulides's Sorites Paradox

The sorites paradox sometimes known as the paradox of the heap is a paradox that arises from vague predicates. Suppose there is a heap of sand, and there are infinitely many individual sand which are removed individually. We assume that removing a single grain does not transform a heap of sand into a non-heap of sand, the paradox is to ask what will it when the process is repeated infinitely and enough times that is a single remaining grain still a heap? If not, when did it change from a heap to a non-heap?

Another famous description of sorites paradox uses the concept of baldness. A man with a full head of hair is obviously not bald. If we remove a single hair, it would not make him bald. By the same reasoning, the removal of two hairs would not make him bald neither. However, a continuation of this process would cause baldness finally".

"In each of these examples, the argument hinges on the presumed 'tolerance' or 'insensitivity' of the key term. That is, a minute difference cannot make a difference, cannot affect the applicability of the predicate" (Olin, pg168). Therefore, this kind of paradox can be constructed for plenty of other predicates: tall, rich, red, small number, and so on.

Many mathematicians have been working on how to solve this kind of paradox. And one way they come up with is to use the which is to deny the most fundamental principle that every statement is either true or false. The fuzzy logic was invented by Dr. Lotfi Zadeh at UC Berkeley. He thinks that every statement could have a degree of true or false instead of being exactly purely true or false. Consider the heap, people could use fuzzy hedges to divide the continuum into small parts corresponding to classes like definitely heap, mostly heap, partly heap, slightly heap, and not heap. Therefore, the sand moves smoothly from “definitely heap” to definitely not heap”, with some intermediate regions.

3. Hilbert’s Grand Hotel Paradox

It is the case that we assume a hotel with infinitely many rooms with infinitely many guests. David Hilbert, a German mathematician, brought up the question to challenge the idea about infinity. He asked that what will the hotel do if there is one more guest coming that day and looking for a place to stay. Hilbert’s idea is that he makes the guest from each room move from their room to the next room,. For example move guest from room 1 to room 2, move guest from room 2 to room 3 and so on. In this way, the new guest will be able to have a room to stay. Then the following question come up. What will happen if there is a bus containing infinitely many guest arriving at the hotel hoping to find rooms to live? Can the hotel comes up with a way that they could accommodate all the guests? Hilbert came up with the solution with the same idea. But the only difference is that he asks the guest from each room to move to the double room number. For example, moves room 1 to 2, moves 2 to 4, moves 3 to 6 and so on. Then all the even number rooms will be the previous guest’s new rooms. And then the odd number rooms could be left empty and all the new guests will be accommodate.

But it is impossible to accommodate infinitely many buses with infinitely many guests on those buses. The hotel manager’s disposable rooms must be countable so in the reality this won’t work.

4.Russel’s Paradox

One of the most famous in the 20th century is Russel’s paradox based on the idea “the set of all sets that do not contain themselves” It is difficult to picture or imagine the situation but it would be easier with a life example. Suppose there is a village and there is only one barber in the village. Everyone in the village needs to cut their hair frequently and no exception. The barbar says “I will only cut hairs for those who does not cut hairs for themselves” So the question is that does the barber cut for himself? If he does not cut for himself, then he satisfy and also qualify the condition he brought up himself, so he has to cut hair for himself. But if he cut hair for himself, that’s a contradiction.

When we ask the question “Does the set R include itself or not?” we have already presupposed that the set R exists. However, what the Russell paradox actually reveals is that it does not. In terms of mathematics, the Russell set R is simple not a well-defined set. “The only real question is why this is so—that is, how it comes about that R’s seemingly straightforward definition goes awry and is illegitimate. It is not R’s demise that is in question, but the result of the autopsy, the question of why it is that R cannot survive a proper scrutiny of its meaningfulness” (Rescher, pg171).

To Dr Z:

I just want to thank you for teaching me this semester and I really enjoy this class. I also learned a lot from this class historically and mathematically. I am graduating this semester and taking one year off. I really hope to take your class again in the future when I go to graduate school.

Again, thank you Dr Z. Hope you all well and have a good summer.

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