Maryam Mirzakhani

Math History Project

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In 2014, Maryam Mirzakhani became the first woman, as well as the first Iranian, to receive the Field Medal in Mathematics. An award often referred to as the Nobel Prize of mathematics. This prize is awarded by the International Mathematical Union every four years to a few brilliant mathematicians under the age of forty (deMelo 2015)."

Mirzakhani was born and raised in Tehran, Iran. As a young girl, she dreamed of becoming a writer. Her favorite pastime was reading novels. These stories gave her the ambition to do something great with her life (Klarreich 2014). She did not decide to pursue mathematics until high school, when her older brother sparked an interest in science. "Her first memory of mathematics is when her brother told her about a problem of adding numbers from one to 100. He had read in a journal how Gauss solved the problem, and the solution was quite fascinating for her (Carlson 2008)."

As she completed elementary school the Iran-Iraq war was ending and opportunities started opening for gifted and driven students. She took a placement test and was placed at the Farzanegan middle school for girls in Tehran, which is administered by Iran's National Organization for Development of Exceptional Talents (Klarreich 2014). In many ways, she was very lucky, she would not have had these great opportunities if she was born ten years earlier (Carlson 2008).

In her first week at middle school she made friends with Roya Beheshti, who is now a mathematics professor at Washington University in St. Louis. They shared many interests, and she helped Mirzakhani stay motivated (Carlson 2008). The two are still friends today. Maryam did poorly in her mathematics class during her first year. Her teacher did not find her to be very talented which negatively affected her confidence. During her next year, Mirzakhani's teacher was more encouraging and her performance improved immensely (Klarreich 2014).

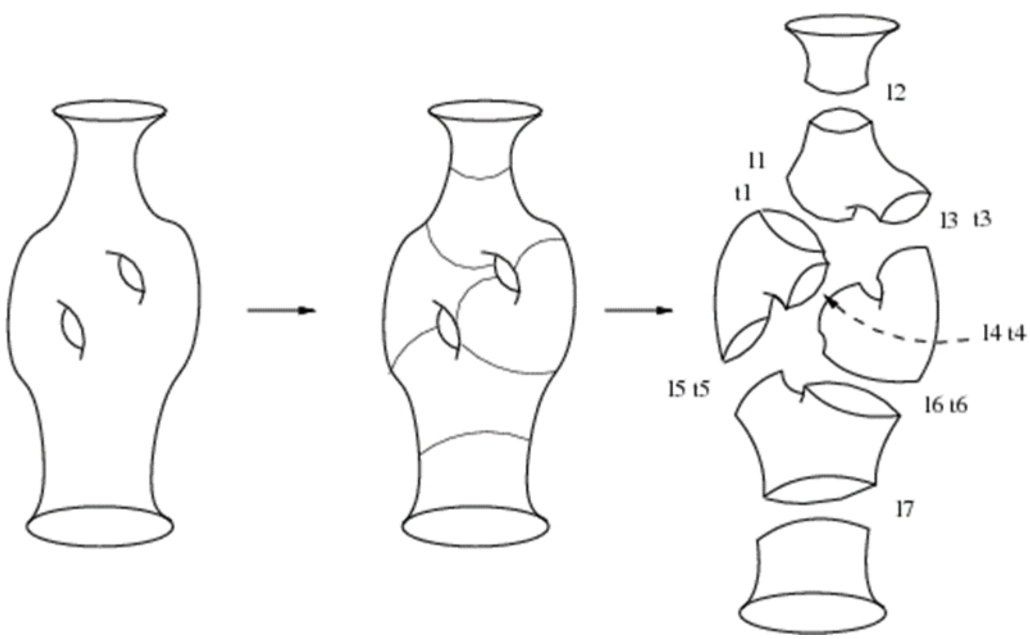
“Mirzakhani went on to the Farzanegan high school for girls. There, she and Beheshti discovered questions from that year’s national competition to determine which high school students would go to the International Olympiad in Informatics, an annual programming competition for high school students (Klarreich 2014).” “Determined to discover how they would perform in similar competitions, Mirzakhani and Beheshti went to the principal of their school and requested that she provide math problem-solving classes similar to the ones being taught at the comparable high school for boys. “The principal of the school was a very strong character,” Mirzakhani recalled. “If we really wanted something, she would make it happen.” The principal was undeterred by the fact that Iran’s International Mathematical Olympiad team had never fielded a girl, Mirzakhani said. “Her mindset was very positive and upbeat, that ‘you can do it, even though you’ll be the first one,” Mirzakhani said. “I think that has influenced my life quite a lot. (Klarreich 2014).”

Mirzakhani first appeared on the international math scene when she was seventeen years old, winning a gold medal at the 1994 International Math Olympiads, and finishing with a perfect score in the 1995 competition (Carey 2014). After graduating with her bachelor’s degree from Sharif University of Technology in 1999, she attended Harvard University and began work on her doctorate under the guidance of Fields Medal recipient Curtis McMullen (Carey 2014).

When she started at Harvard, her background was mostly in combinatorics and algebra. Then, she began attending an informal seminar organized by McMullen. She started asking questions consistently, and thinking about problems that would come up during these discussions. By the time she left Harvard, she had a substantial list of ideas that she wanted to explore (Carlson 2008).

Mirzakhani became interested in hyperbolic surfaces, doughnut-shaped surfaces with two or more holes that have a non-standard geometry which, gives each point on the surface a saddle shape (Klarreich 2014). When she started at Harvard, some of the simplest questions about these types of surfaces were unanswered. One concerned straight lines, or “geodesics,” on a hyperbolic surface. A curved surface can have an idea of a “straight” line segment: it’s simply the shortest course between two points. On a hyperbolic surface, some geodesics are infinite in length, but others close into a loop. Mirzakhani’s work concerned closed geodesics, or closed curves whose length cannot be shortened by deforming them (Work 2014).

A theorem that was proven over five decades ago, the “prime number theorem of geodesics” gives an accurate method of estimating the number of closed geodesics whose length is smaller than some arbitrary bound L. Mirzakhani noticed what happens to the “prime number theorem for geodesics” when one considers only the closed geodesics that are simple, meaning that they do not intersect themselves (Work 2014).



"In Mirzakhani's proof of her counting result for simple closed geodesics, another structure on moduli space enters, a so-called symplectic structure, which, allows one to measure volumes, but not lengths (Work 2014). Mirzakhani discovered a remarkable generalization of McShane's Identity. Theorem (McShane):

Let $f(x) = (1 + e x) - 1$ and let X be a hyperbolic torus with a cusp.

Then, $\sum_{\gamma} f\left(\ell_{\gamma}(X)\right) = \frac{1}{2}$,

where the sum is taken over all simple closed geodesics γ on X,

and $\ell_{\gamma}(X)$ is the length of the geodesic γ (de Melo 2015)."

"Mirzakhani's more general identity does not immediately yield the volume. However, cutting the initial surface by simple closed geodesics involved in her identity and developing the idea of averaging over all possible hyperbolic surfaces, Mirzakhani gets a recursive relation for the volume Vg, n(L1, . . . , Ln) := Vol Mg, n(L) in terms of analogous volumes of simpler moduli spaces. These relations allow Mirzakhani to prove the following statement and to compute the volumes explicitly (de Melo 2015)." Theorem ([M2]):

The volume Vg, n(L1, . . . , Ln) is a polynomial in L 2 1 , . . . , L2 n ; namely, we have

$$Vg, n(L) = \sum_{|\alpha| \leq 3g-3+n} C\alpha \cdot L\, 2\alpha \text{, where } C\alpha > 0 \text{ lies in } \pi^{6g-6+2n-2|\alpha|}$$

This established a connection between the volume calculations on moduli space and the counting problem for simple closed geodesics on a single surface. She determines certain volumes in moduli space and then calculates the counting result for simple closed geodesics from this calculation (Work 2014). Another consequence of this work was a surprising new proof of a conjecture proposed by physicist Edward Witten of the Institute for Advanced Study in

Princeton, NJ, about certain topological measurements of moduli spaces relating to string theory (Klarreich 2014). Mirzakhani's doctoral thesis produced three papers that were published in the three top journals of mathematics: Annals of Mathematics, Inventiones Mathematicae and Journal of the American Mathematical Society (Klarreich 2014) .

From 2004 to 2008, she was a Clay Mathematics Institute Research Fellow and an assistant professor at Princeton University. According to Mirzakhani, "It was a great opportunity for me. I spent most of my time at Princeton which was a great experience. The Clay Fellowship gave me the freedom to think about harder problems, travel freely, and talk to other mathematicians. I am a slow thinker, and have to spend a lot of time before I can clean up my ideas and make progress. So, I really appreciate that I didn't have to write up my work in a rush (Carlson 2008)."

Unlike some mathematicians who solve problems very quickly, she is drawn toward deep problems that she can work on for years. "Months or years later, you see very different aspects" of a problem, she said (Klarreich 2014).

As she thinks about a problem, Mirzakhani constantly doodles, drawing surfaces and other images related to her research. "She has these huge pieces of paper on the floor and spends hours and hours drawing what look to me like the same picture over and over," stated Jan Vondrak, Mirzakhani's husband, and theoretical computer scientist at IBM Almaden Research Center in San Jose, CA (Klarreich 2014).

When thinking about a difficult math problem, "you don't want to write down all the details," she said. "But the process of drawing something helps you somehow to stay connected."

Mirzakhani said that her 3-year-old daughter, Anahita, often exclaims, “Oh, Mommy is painting again!” when she sees the mathematician drawing (Mehta 2016).

In 2006, she began a beneficial collaboration with Alex Eskin, a mathematics professor at the University of Chicago (Klarreich 2014). In 2008, she became a Professor of mathematics at Stanford, where she still lives with her husband and daughter (Carey 2014).

After several projects together, Mirzakhani and Eskin decided to tackle one of the largest open problems in their field. This concerned the range of behaviors of a ball that is bouncing around a polygon shaped billiard table, as long as the degrees of the angles are a rational number. Billiards gives some simple examples of dynamical systems, or systems that evolve over time according to a given set of rules, but the behavior of the ball has proven exceedingly hard to compute (Klarreich 2014).

“Rational billiards got started a century ago, when some physicists were sitting around saying, ‘Let’s understand a billiard ball bouncing in a triangle,” said Alex Wright, a postdoctoral researcher at Stanford. “Presumably, they thought they would be done in a week, but 100 years later, we’re still thinking about it (Klarreich 2014).”

“By transforming each billiard table into an abstract surface called a translation surface, mathematicians can analyze billiard dynamics by understanding the larger moduli space consisting of all translation surfaces (Klarreich 2014).” Researchers have found that understanding the orbit of a translation surface as the squishing action moves it around in the moduli space helps in solving many questions about the original billiard table. On the surface, this orbit might be an extremely complex object, a fractal, for example. However, in 2003 McMullen showed that this is not so when the translation surface is a two-holed (“genus two”) doughnut. Every single orbit fills up either the entire space or some simple subset of the space called a submanifold (Klarreich 2014).

After years of work, in 2012 and 2013, Mirzakhani and Eskin, with some help from Amir Mohammadi of the University of Texas at Austin, succeeded in generalizing McMullen's result to surfaces of any genus, or doughnut surfaces with more than two holes (Klarreich 2014). Their analysis is "a titanic work," Zorich said, adding that its implications go far beyond billiards. The moduli space "has been intensively studied for the last 30 years," he said, "but there's still so much we don't know about its geometry (Klarreich 2014)." This was breakthrough in understanding another dynamical system on moduli space that is related to the behavior of geodesics in moduli space (Work 2014).

Non-closed geodesics in moduli space are very unpredictable, and it is hard to grasp any understanding of their structure and how they change when disturbed slightly. However, Mirzakhani along with Eskin and Mohammadi have proved that complex geodesics and their closures in moduli space are in fact surprisingly regular, rather than irregular or fractal (Work 2014). While complex geodesics are transcendental objects defined in terms of analysis and differential geometry, their closures are algebraic objects defined in terms of polynomials and therefore have certain rigidity properties (Work 2014).

Their work has received recognition among researchers in the area, who are working to expand and build on the new result. One reason the work ignited so much interest, is that the theorem Mirzakhani and Eskin proved is comparable to a celebrated result of Marina Ratner from the 1990s (Work 2014). "Ratner established rigidity for dynamical systems on homogeneous spaces, these are spaces in which the neighborhood of any point looks just the same as that of any other point. By contrast, moduli space is totally inhomogeneous: Every part of it looks totally different from every other part. It is astounding to find that the rigidity in homogeneous spaces has an echo in the inhomogeneous world of moduli space (Work 2014)."

Mirzakhani has big plans for the future. She has started working with Wright to develop a complete list of the type of sets that translation surface orbits can fill. Such a classification would be a "magic wand" for understanding billiards and translation surfaces, Zorich has written (Klarreich 2014)."

Many times, research into areas like this have unexpected applications, but that isn't what motivates Mirzakhani to pursue them. Instead, the motivation is to understand, as deeply as possible, these basic mathematical structures, said Ralph Cohen, a professor of mathematics and the senior associate dean for the natural sciences in Stanford's School of Humanities and Sciences. "Maryam's work really is an outstanding example of curiosity-driven research (Carey 2014)." However, this work could have impacts concerning the theoretical physics of how the universe came to exist and, because it could inform quantum field theory, secondary applications to engineering and material science. Within mathematics, it is important to the study of prime numbers and cryptography. (Carey 2014).

In spite on these important applications, Mirzakhani says she enjoys mathematics because of the elegance and longevity of the questions she studies. "I don't have any particular recipe," Mirzakhani said of her approach to developing new proofs. "It is the reason why doing research is challenging as well as attractive. It is like being lost in a jungle and trying to use all the knowledge that you can gather to come up with some new tricks, and with some luck you might find a way out (Carey 2014)." Mirzakhani is a brilliant mathematician, and has broken the glass ceiling for many women in her field. Hopefully, we will see many more women recognized for their contributions to mathematics in the future.

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