MATH 436 (Spring 2017), Solutions to Exam 0, Thurs., Jan. 19, 2017 (SEC 211), 11:00-11:40am

1. Write seventeen in (a) base 2 (b) base 3

Ans.(a): seventeen is expressed as: (in base 2) 10001

Ans.(b): seventeen is expressed as: (in base 3) 122

Solution of 1(a) In binary notation the "building blocks" are the powers of 2: $1, 2, 4, 8, 16, 32, \ldots$. We look at the highest power of 2 that is ≤ 17. It is 16. So

$$17 = 16 + 1 \quad .$$

We now have to do the same for 1, but this is already a building block. So the expression of 17 as a sum of powers of 2 is

$$17 = 2^4 + 2^0 = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = (10001)_2$$

Solution of 1(b) In base 3 notation the "building blocks" are the powers of 3: $1, 3, 9, 27, \ldots$. But now each building block can show up one or two times. We look at the **last** power of 2 less than 17, it is 9. Since 17 is less than twice 9, 9 only shows up once, so we writ

$$17 = 1 \cdot 3^2 + 8 \quad .$$

We now have to handle 8. The highest power of 3 less than 8 is 3, but twice 3, is less than 8, so we write

$$8 = 2 \cdot 3 + 2$$

Finally

$$2 = 2 \cdot 1 \quad .$$

Combining, we have

$$17 = 1 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0 = (122)_3 \quad .$$

Comments: **1.** About 12 out of 24 people got them both. Most people got the (a) right. We will cover this topic in class later on. It is very basic (and historical!)
2. Some people misunderstood what 'writing a number in base 2' means and understood it as base of a logarithm.

2 Prove that $\sqrt{2}$ is irrational.

Sol. of 2: Suppose not, then $\sqrt{2} = \frac{a}{b}$ where a and b are positive integers. We can assume that they are not both even (if they are, you can keep dividing top and bottom by 2 until at least one of them is odd).

Hence $2 = \frac{a^2}{b^2}$. Hence $a^2 = 2b^2$ hence a^2 is even, hence a is even (if a is odd then a^2 is also odd (why?)). Hence we can write

$$a = 2c \quad ,$$

for some other integer c. Hence

$$(2c)^2 = 2b^2 \quad ,$$

Hence

$$2c^2 = b^2 \quad ,$$

hence b is even. So a and b are **both** even, while we assumed that at least one of them is odd. Contradiction.

Comments: **1.** Only 4 out of 24 people got it completely! A few other people started it correctly, but then got stuck. Some people wrote gibberish. It is more important to admit that you don't know than write gibberish.

2. This is a **VERY IMPORTANT** result, historically speaking. We will disuss it in class, later on, and present a much nicer proof.

3. State and prove the Pythagorean Theorem.

Solutions: See `https://en.wikipedia.org/wiki/Pythagorean_theorem` .

Comment: 20 out of 24 people got the statement right. Only 4 out of 24 people got the proof right. We will discuss at least three proofs in the class.

4. I am a positive integer. If you divide me by 3, you get remainder 2. If you divide me by 7, you get remainder 6. I am as small as can be (under the above conditions). Who am I?

Ans. to 4: 20.

Sol. to 4: This is a special case of the famous **Chinese Remainder Theorem**, that we will discuss in class. The fastest way is to list all the integers that give you remainder 6 when you divide them by 7:

$$6, 13, 20, \ldots,$$

and look at the remainder after dividing them by 3

$$0, 1, 2, \ldots$$

so 20 is the smallest answer.

Comments: **1.** 22 out of 24 people got it right.

2. I hope to teach an algorithm for doing it in general.

5. State an explicit expression for

$$\sum_{i=1}^{n} i^2$$

Ans.

$$\frac{n(n+1)(2n+1)}{6} \quad .$$

Comments: **1.** Only **one** student got it right! When I was your age, *every* mathematics student knew this. Times have changed...

2. We will learn later on how to derive such a formula from scratch.

6. State an explicit expression for

$$\sum_{i=1}^{n} i^3$$

Ans. to 6

$$\left(\frac{n(n+1)}{2}\right)^2 \quad .$$

Comments: **1.** 4 students got it right.

7. The sequence of Fibonacci numbers, F_n, are defined by $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$, by

$$F_n = F_{n-1} + F_{n-2} \quad .$$

What is F_7?

Ans. to 7: $F_7 = 13$.

Comments: **1.** 21 students (out of 24) got it right.
8. With F_n defined as above, can you conjecture a nice expression for A_n defined by

$$A_n := F_{n+1}F_{n-1} - F_n^2 \quad .$$

Ans. to 8: $A_n = (-1)^n \quad .$

Comments: **1.** 7 students (out of 24) got it right. This was a tricky questions, congratulations to those who got it.

9 What are the usual names for the following functions, given in terms of their power series expansions

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad ,$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \quad ,$$

$$\sum_{n=0}^{\infty} x^n \quad .$$

Ans. to 9: $\cos x$, $\sin x$, e^x, $\frac{1}{1-x}$, respectively.

Comments: **1.** No one got the last one (one student came close and wrote $\frac{1}{1+x}$). Only 4 students (out of 24) got all the first three right. (Two students interchanged the sine and cosine). 12 students got the e^x right, but I mentioned it in class!

10. Order the following famous mathematicians according to their year of birth, from oldest (most ancient) to the youngest (most modern).
Gauss, Archimedes, Zeilberger, Euler, Hilbert, Laplace, Galois, Cayley .
For each, state the century of birth.

Ans. to 10: Archimedes (287 BC), Euler (1707), Laplace (1749), Gauss (1777), Galois (1811), Cayley (1821), Hilbert(1862), Zeilberger (1950).

Comment: No one got the exact ordering right, but some were close.

11. What does Fermat's Last Theorem Claim? Who proved it?

Ans. to 11: If $n \geq 3$, then there are no positive integers x, y, z such that $x^n + y^n = z^n$. Andrew Wiles (with the help of Richard Taylor) first proved it.

Comments: Only 3 students got the statement right. **No one** has ever heard of Sir Andrew Wiles. I feel bad for him. He should be more famous than a rock star!

12. What is the Riemann Zeta Function, $\zeta(s)$?. What Is the Riemann Hypothesis?

Ans. to 12: See
`https://en.wikipedia.org/wiki/Riemann_zeta_function`

Comment: No one got it right. This is an advanced topic, **but** it is the most famous open problem in mathematics!

13. List the first 10 prime numbers.

Ans. to 13: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29$

Comment: 19 (out of 24) students got it right. A few people included 1 (1 is **not** prime!), and another student even included 0 (0 is definitely **not** a prime, it is divisible by everthing!)

14. Prove that there are infinitely many primes.

Sol. to 14: Suppose that there are only finitely many of them, and let their number be n, where n is some (finite) integer. let's order them

$$p_1, p_2, \ldots, p_n \quad .$$

Consider a brand-new number

$$P = p_1 p_2 \cdots p_n + 1 \quad .$$

Then P leaves remainder 1 when divided by each of the primes $p_1, \ldots, p_n$, but every integer must be divisible by *some* prime, so this is absurd. Hence it is not true that there are finitely many primes.

Comments: **1.** Only 3 (out of 24) students gave a completely correct proof. Some people came close, but some did total nonsense. It is better to admit that you don't know than trying to fake it.

15. The perfect Platonic solids are the tetrahedron (four faces), cube (six faces), octahedron (eight faces), dodecahedron (12 faces), and icosahedron (20 faces). Do you know of a relation between the number of vertices, V, the number of edges, E, and the number of faces, F? (For example, for a cube $V = 8$, $E = 12$, $F = 6$). Who discovered this relation?

Ans. to 15:

$$V - E + F = 2 \quad ; \quad Euler \quad .$$

Comments: 1. 4 (out of 24 students) got the formula right. 6 got Euler right. **2.** This is a historically important formula, and I hope to cover it in class (much later).

16. Is it possible to construct a square with the same area of a given circle, only using straight-edge and compass?

Ans. to 16: **NO**.

Comments 1. Only 5 (out of 24 students) got it right. **2.** This is one of the famous ancient Greek problems. We will talk about it in class in due course.

17. If you toss a fair coin 10 times, what is the probability of getting at most two Heads?

Ans. to 17

$$\frac{1}{2^{10}}\left(\binom{10}{0}+\binom{10}{1}+\binom{10}{2}\right)=\frac{56}{1024}=\frac{7}{128} \quad .$$

Comments: **1.** Only 4 out of 24 students got it right. **2.** Basic probability is very important historically. I hope to discuss it in class in due course.