Solutions of the Attendance Quiz # 4 for Dr. Z.'s MathHistory for Lecture 4

1. Two bike-riders, A and T have a race. A rides at a constant speed of 32 miles per hour, while T is much slower, and rides at a constant speed of 16 miles per hour.

To make it fair, T gets a head-start of 16 miles.

(a) Spell out Zeno's "proof" that A will **never** catch-up to T.

Sol. to (a):

At the beginning $t = 0$,

A's distance from the start is: 0 T's distance from the start is: 16

It takes A $16/32 = \frac{1}{2}$ hour to get to T's starting point, but by then T made it to $16 + \frac{1}{2} \cdot 16 = 16 + 8 = 24$ so

At time $t = \frac{1}{2}$ hours:

A's distance from the start is: 16 T's distance from the start is: 24

At time $t = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ hours:

A's distance from the start is: 24 T's distance from the start is: 28

Continuing, the same way:

At time $t = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$ hours

A's distance from the start is: 28 T's distance from the start is: 30

At time $t = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$ hours

A's distance from the start is: 30 T's distance from the start is: 31

At time $t = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}$ hours

A's distance from the start is: 31 , T's distance from the start is: $31\frac{1}{2}$

At time $t = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = \frac{63}{54}$ hours

A's distance from the start is: $31\frac{1}{2}$ T's distance from the start is: $31\frac{3}{4}$

and so on, and so forth. Whenever A will reach the point that T was in the previous step, T would have gotten a bit further, *ad infinitum*, so (according to Zeno), A will never catch up to T!

(b) Use high-school algebra (the formula that distance=speed multiplied by time) to predict at what time, in spite of Zeno, A will catch-up with T after all.

Sol. to (b):

After t hours, the distance of A from the starting point is $32t$ miles, and the distance of T from the starting point is $16 + 16t$. The meeting time is obtained by setting these two expressions **equal** to each other (when they meet, their distance from the staring point is **equal**). So we have to solve the algebraic equation

$$32t = 16 + 16t \quad .$$

Subtracting $16t$ from both sides gives

$$16t = 16 \quad .$$

Dividing by 16, we get

$$t = 1 \quad .$$

Ans. to (b): According to high-school algebra, (and common sense!), A and T **do** meet after exactly one hour.

(c) Express the meeting time as an 'infinite' geometric series, and use the formula for summing an 'infinite' geometric series to get the same answer as (b).

Sol. to (c) Adding up the successive time-intervals needed for A to catch up with T's previous location we have

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \frac{1}{2} \cdot (1 + \frac{1}{2} + \frac{1}{4} + \ldots) = \frac{1}{2} \sum_{n=0}^{\infty} (\frac{1}{2})^n \quad .$$

Using the **famous** formula (only valid if $|x| < 1$, but this is OK, since $x = \frac{1}{2}$)

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad ,$$

we get that the meeting time is

$$\frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{\frac{1}{2}} = 1 \quad .$$

Ans. to (c): The meeting time, by summing the 'infinite' geometrical series is 1 (same as using the other way in (b)!).

(d) If, like Dr. Z. (and much greater people, like Hermann Weyl), you dislike 'infinite' sums, resolve the paradox, by assuming that the 'atom' (smallest unit) of time is $\frac{1}{32}$ of an hour. Compile a table of each successive step in the Zeno description of the race, but rather than going for ever, stop when it is no longer possible to sub-divide time.

Sol. to (d)

The trouble starts after the following stage

At time $t = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}$ hours

A's distance from the start is: 31 $\quad$ T's distance from the start is: $31\frac{1}{2}$

The next step requires $\frac{1}{64}$ seconds, but this is **meaningless**. Time advances by 'atoms' of $\frac{1}{32}$ hours.

To the next step is

At time $t = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{32} = 1$ hours

A's distance from the start is: 31+1=32 $\quad$ T's distance from the start is: $31\frac{1}{2} + \frac{1}{2} = 32$.

They did catch up! Yea to the **fact** that time is discrete!

Note: According to modern physics, there is an 'atom of time', it is the time it takes light to travel the 'atom of distance', the so called 'Planck's length', about, $1.616 \cdot 10^{-35}$ meters. Since the speed of light is about about $3 \cdot 10^8$ meters per second, the 'Plank time unit' is about

$$\frac{1.616 \cdot 10^{-35}}{3 \cdot 10^8} = 8.08 \cdot 10^{-42} \; seconds$$

Of course, much smaller than our fictional $\frac{1}{32}$ hours in the problem, but the same resolution of Zeno's paradox still holds, only one has to go many more steps.