**Solutions to the Attendance Quiz # 3 for Dr. Z.'s MathHistory for Lecture 3**

**1.** Use Dr. Z.'s method to conjecture an explicit formula for

$$\sum_{i=1}^{n} \frac{i(i-1)}{2} \quad .$$

Prove it

(a) The 'usual' (boring way)

(b) the fun, Dr. Z. way.

**Sol. 1**: The first step is to come up with an expression. Since the *summand* is of degree 2, the quantity, let's call it $A(n)$,

$$A(n) = \sum_{i=1}^{n} \frac{i(i-1)}{2} \quad ,$$

is *some* polynomial of degree 3. So we write down a template

$$A(n) = an^3 + bn^2 + cn + d \quad ,$$

where $a, b, c, d$ are **to be determined**.

First, let's *collect data*.

$$A(0) = 0 \quad , \quad A(1) = (1)(0)/2 = 0 \quad , \quad A(2) = (1)(0)/2 + (2)(1)/2 = 0 + 1 = 1 \quad ,$$
$$A(3) = (1)(0)/2 + (2)(1)/2 + (3)(2)/2 = 0 + 1 + 3 = 4 \quad .$$

On the other hand

$$A(0) = a \cdot 0^3 + b \cdot 0^2 + c \cdot 0 + d = d \quad ,$$
$$A(1) = a \cdot 1^3 + b \cdot 1^2 + c \cdot 1 + d = a + b + c + d \quad ,$$
$$A(2) = a \cdot 2^3 + b \cdot 2^2 + c \cdot 2 + d = 8a + 4b + 2c + d \quad ,$$
$$A(3) = a \cdot 3^3 + b \cdot 3^2 + c \cdot 3 + d = 27a + 9b + 3c + d \quad ,$$

Equating we get a system

$$d = 0 \quad , \quad a + b + c + d = 0 \quad , \quad 8a + 4b + 2c + d = 0 \quad , \quad 27a + 9b + 3c + d = 0$$

We get right away that $d = 0$, so we have to solve

$$a + b + c = 0 \quad , \quad 8a + 4b + 2c = 0 \quad , \quad 27a + 9b + 3c = 0$$

This is a bit tedious, but when you do it, you get

$$a = \frac{1}{6} \quad , \quad b = 0 \quad , \quad c = -\frac{1}{6} \quad , \quad d = 0$$

This gives

$$A(n) = \frac{1}{6}n^3 + 0 \cdot n^2 - \frac{1}{6}n + 0 = \frac{(n+1)n(n-1)}{6} \quad .$$

This ends the **first part**.

**Second Part**: Proving.

Dr. Z.'s way

$$n = 0 \quad : \quad EmptySum = 0 = \frac{(0+1)0(0-1)}{6} \quad (yea!)$$

$$n = 1 \quad : \quad (0)(1)/2 = \frac{(1+1)1(1-1)}{6} \quad (yea!)$$

$$n = 2 \quad : \quad (0)(1)/2 + (1)(2)/2 = \frac{(2+1)2(2-1)}{6} \quad (yea!)$$

$$n = 3 \quad : \quad (0)(1)/2 + (1)(2)/2 + (2)(3)/2 = \frac{(3+1)3(3-1)}{6} \quad (yea!)$$

**QED** (since both sides are polynomials of degree three and they argee at four different values).

**Official(Boring Way)**:

We have to prove, for *every* non-negative integer $n$, that

$$\sum_{i=1}^{n} \frac{i(i-1)}{2} = \frac{(n-1)n(n+1)}{6} \quad .$$

Base case $n = 0$: $0 = 0$. OK.

**Inductive Hypothesis** (replace $n$ by $n - 1$)

$$\sum_{i=1}^{n-1} \frac{i(i-1)}{2} = \frac{(n-2)(n-1)n}{6} \quad .$$

By the definition of summation (pulling out the last term):

$$\sum_{i=1}^{n} \frac{i(i-1)}{2} = \left( \sum_{i=1}^{n-1} \frac{i(i-1)}{2} \right) + \frac{n(n-1)}{2}$$

By the inductive hypothesis this equals

$$\frac{(n-2)(n-1)n}{6} + \frac{n(n-1)}{2} \quad .$$

Simplifying this equals

$$\frac{(n-1)n}{6}((n-2)+3) = \frac{(n-1)n}{6}(n+1) = \frac{(n-1)n(n+1)}{6} \quad .$$

**QED**.

**Yet another (semi-boring proof)** (suggested by one student)

$$\sum_{i=1}^{n} \frac{i(i-1)}{2} = \sum_{i=1}^{n} \left( \frac{1}{2}i^2 - \frac{1}{2}i \right) = \frac{1}{2}\sum_{i=1}^{n} i^2 - \frac{1}{2}\sum_{i=1}^{n} i \quad .$$

We mentioned, and proved, in class, the famous identities

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad ,$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad .$$

So going back, we have

$$\sum_{i=1}^{n} \frac{i(i-1)}{2} = \frac{1}{2}\sum_{i=1}^{n} i^2 - \frac{1}{2}\sum_{i=1}^{n} i = \frac{1}{2}\frac{n(n+1)(2n+1)}{6} - \frac{1}{2}\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{12}\left((2n+1)-3\right) = \frac{n(n+1)}{12}\left(2n-2\right)$$

$$= \frac{n(n+1)}{12} \cdot 2 \cdot (n-1) = \frac{(n-1)n(n+1)}{6} \quad .$$

QED.

**Comment**: This problem is very long, and I did not allow enough time. Congratulations to (quite a few!) people who did it completely.