**Solutions to Attendance Quiz # 18 for Dr. Z.'s MathHistory Lecture 18**

**1.** Prove that $\sqrt{17}$ is irrational.

**Sol.**

Lemma: Every integer $n$ can be written as $17^i a$ where $gcd(a, 17) = 1$, for some integer $i \geq 0$

(you can keep dividing by 17 until you can'd to it any more).

Assume that $\sqrt{17} = \frac{m}{n}$ where $m$ and $n$ are positive integers. By squaring

$$17 = \frac{m^2}{n^2} \quad ,$$

hence

$$m^2 = 17n^2 \quad .$$

By the lemma $m = 17^i a$ and $n = 17^j b$ for some non-negative integers $i$ and and $j$, hence

$$(17^i a)^2 = 17(17^j b)^2 \quad ,$$

Hence

$$17^{2i} a^2 = 7^{2j+1} b^2 \quad ,$$

So the power of 17 in the prime-decomposition of the left is an even integer, while on the right it is an odd integer. Since they are supposed to be equal, we arrived at a contradiction!

**2.**

Starting at the initial position of the 9-puzzle

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & \end{pmatrix}$$

determine Which of the following positions can **never** be reached by sliding. Explain!

(a)

$$\begin{pmatrix} 8 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 1 & \end{pmatrix}$$

(b)

$$\begin{pmatrix} & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 1 \end{pmatrix}$$

**Sol.** We use the quantity

$$S = inv(\pi) + i + j$$

where $\pi$ is the permutation correponding to the position, where the blank is replaced by 9, and $i$ and $j$ are the row- and column number of the blank.

For (a), $\pi = 823456719$, $i = 3$, $j = 3$. We have

$$inv(\pi) = 7 + 1 + 1 + 1 + 1 + 1 + 1 = 13 \quad ,$$

and

$$S = 13 + 3 + 3 = 19 \quad ,$$

Since this is **odd** (and the $S$ for the initial position is even) this position is **impossible**.

For (b), $\pi = 923456781$, $i = 1$, $j = 1$. We have

$$inv(\pi) = 8 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 15 \quad ,$$

So

$$S = 15 + 1 + 1 = 17$$

Once again $S$ is **odd** (and the $S$ for the initial position is even) this position is **impossible**.