Solutions to MATH 436 Exam I for Dr. Z.'s Math History Course Spring 2017, March 23, 2017

1. (10 pts.) Prove that there are infinitely many primes.

Sol. Suppose that there are only finitely many of them, say k of them, and let's call them $p_1, \dots, p_k$. Let's create a big integer

$$P = p_1 \cdot p_2 \cdots p_k + 1 \quad .$$

P, being a positive integer, must be either a prime itself, or divisibile by at least one prime.

Note that

• It is **not divisible** by p_1, since when you divide P by p_1 you get remainder 1

• It is **not divisible** by p_2, since when you divide P by p_2 you get remainder 1

...

• It is **not divisible** by p_k, since when you divide P by p_k you get remainder 1

So it must be divisible (or be itself) by yet another prime **none of the above**. So we found another prime! This contradicts the assumption that $p_1, \dots, p_k$ are the **only** primes in the world. So whenever you think that you have found all the primes, you can always come up with yet-another-one, hence there are infinitely many of them.

2. (10 pts.) Prove that $\sqrt{5}$ is irrational.

Sol.

First Proof: We first prove a

Lemma: If n is **not** divisible by 5 then n^2 is also **not** divisible by 5.

Proof of Lemma: We can write $n = 5m + 1$, or $n = 5m + 2$ or $n = 5m + 3$ or $n = 5m + 4$.

$(5m + 1)^2 = 25m^2 + 10m + 1 = 5(5m^2 + 2m) + 1$

$(5m + 2)^2 = 25m^2 + 20m + 4 = 5(5m^2 + 4m) + 4$

$(5m + 3)^2 = 25m^2 + 30m + 9 = 5(5m^2 + 6m + 1) + 4$

$(5m + 4)^2 = 25m^2 + 40m + 16 = 5(5m^2 + 8m + 3) + 1$

So none of them is divisible by 5.

Corollary: If n^2 is divisible by 5 then n is divisible by 5.

Proof that $\sqrt{5}$ is irrational:

Suppose, for the sake of argument, that $\sqrt{5}$ can be written as

$$\sqrt{5} = \frac{m}{n} \quad ,$$

where m and n are both positive integers. If m and n are both divisible by 5, we can cancel out 5 until at least one of them is not divisible by 5.

So if they exist a pair of positive integers m and n such that $\sqrt{5} = \frac{m}{n}$, then there also exist a pair of integers (let's call them again m and n) such that $\sqrt{5} = \frac{m}{n}$, and m and n are **not both divisible by 5**.

Squaring both sides

$$5 = \frac{m^2}{n^2} \quad .$$

By algebra

$$m^2 = 5n^2 \quad .$$

Hence m^2 is divisible by 5, it follows from the corollary to the lemma that m is divisible by 5, hence we can write

$$m = 5a \quad ,$$

for *some* integer a.

Hence

$$(5a)^2 = 5n^2 \quad .$$

By algebra

$$25a^2 = 5n^2 \quad ,$$

More algebra

$$n^2 = 5a^2 \quad ,$$

hence, by the corollary to the lemma, n is divisible by 5. So *both* m and n are divisibile by 5, contradictiong the assumption that m are **not** both divisible by 5. Hence we have to renounce the assertion that $\sqrt{5}$ can be written as $\frac{m}{n}$ for positive integers m and n.

A Second Proof.

First prove the lemma that every positive integer n can be written *uniquely* as

$$n = 5^i m \quad ,$$

for i a non-negative integers and m **not divisible** by 5.

Proof of Lemma: Keep dividing n by 5 until you get an integer not divisible by 5 (possibly 1). To prove uniqueness, suppose $5^i m = 5^j m'$ (where neither m nor m' are divisible by 5) If $i > j$ then m' is divisible by $5^(i-j)$, hence by 5, and if $i < j$ then m is divisible by $5^(j-i)$ hence by 5. Contradiction.

So every integer n can be written in the **format**

$$n = 5^i \cdot m$$

with i a non-negative integer and m **not** divisible by 5.

Hence any integer of the form n^2 can be written as

$$n^2 = 5^{2i} \cdot m \quad ,$$

for *some* i,

and any integer of the form $5n^2$ can be written as

$$5n^2 = 5^{2j+1} \cdot m \quad ,$$

for *some* j.

So the set of integers of the form n^2 *always* have an *even* exponet in their 5-decomposition and those of the form $5n^2$ *always* have an *odd* exponet in their 5-decomposition, hence they can **never overlap**, so $5n^2 = m^2$ can never happen.

3. (10 pts.) Prove that

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2n+1} \quad .$$

Sol.

By calculus, $(\arctan x)' = \frac{1}{1+x^2}$, hence

$$\arctan x = \int_0^x \frac{1}{1+t^2}\, dt \quad .$$

Recall the famous *infinite geometric series* (valid for $|w| < 1$)

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n \quad .$$

Plugging-in $w = -t^2$, we get

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-t^2)^n = \sum_{n=0}^{\infty} (-1)^n t^{2n} \quad .$$

Integrating, *term-by-term*

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n \int_0^x t^{2n} = \sum_{n=0}^{\infty}(-1)^n \left(\frac{t^{2n+1}}{2n+1}\right)\Big|_0^x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} \quad .$$

QED.

4. (10 pts.) Express

$$a(n) := \sum_{k=1}^{n} k(k-1) \quad ,$$

as a polynomial of degree three in n. Prove it!

First Sol.

The summand has degree 2 in k, hence the **sum**, $a(n)$ has degree $2+1=3$ in n, i.e. must be a **polynomial** of degree 3 in n.

Let's collect some data

$$a(0)=0 \quad , \quad a(1)=0 \quad , \quad a(2)=1\cdot 0+2\cdot 1=2 \quad , \quad a(3)=1\cdot 0+2\cdot 1+3\cdot 2=8 \quad ,$$

Since $a(0)=0$ and $a(1)=0$, $a(n)$ is divisible by $n(n-1)$, hence we can write

$$a(n) = n(n-1)(An+B) \quad ,$$

where A and B are **to be determined**.

Since $a(2)=2$, we have

$$2\cdot(2-1)(2A+B)=2 \quad ,$$

so $2A+B=1$.

Since $a(3)=2$ we have

$$3\cdot 2(3A+B)=8,$$

so $3A+B=\frac{4}{3}$. Hence we have to solve the system of two equations and two unknowns (A and B)

$$2A+B=1 \quad , \quad 3A+B=\frac{4}{3} \quad .$$

Subtracting the second equation from the first, we get $A=\frac{4}{3}-1=\frac{1}{3}$. Hence $B=1-2A=\frac{1}{3}$. So $A=\frac{1}{3}$ and $B=\frac{1}{3}$. So we got

$$a(n) = n(n-1)(\frac{1}{3}n+\frac{1}{3}) = \frac{(n-1)n(n+1)}{3} \quad .$$

Ans. to 4: $a(n) = (n-1)n(n+1)/3$.

Note: This is a fully rigorous proof, since both sides are polynomials of degree 3 and they coincide in **four** distinct places, i.e. $n = 0, 1, 2, 3$.

Second Solution

$$a(n) := \sum_{k=1}^{n} k(k-1) = \sum_{k=1}^{n} (k^2 - k) = \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k \quad .$$

Using the famous formulas

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad ,$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad ,$$

we get

$$a(n) = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = \frac{n(n+1)}{6} \cdot (2n+1-3) = \frac{n(n+1)}{6} \cdot (2n-2)$$

$$= \frac{n(n+1)}{6} \cdot (2(n-1)) = \frac{(n-1)n(n+1)}{3} \quad .$$

5. (10 points) Construct a seven by seven Magic Square.

Ans. to 5:

4	35	10	41	16	47	22
29	11	42	17	48	23	5
12	36	18	49	24	6	30
37	19	43	25	7	31	13
20	44	26	1	32	14	38
45	27	2	33	8	39	21
28	3	34	9	40	15	46

6. (10 points) Arrange the following people according to their year-of-birth, from oldest to youngest.

Newton, Archimedes, Gallileo, Euler, Gauss, Zeilberger, Euclid, Thales, Brahmagupta, Fibonacci.

For each person, state their century of birth.

Ans. to 6:

Thales: sixth century BC

Euclid: fourth century BC

Archimedes: third century BC (more precisely 287 BC)

Brahmagupta: seventh century

Fibonacci: late 12th century

Galilleo: late 16th

Newton: 17th century (1642)

Euler: 18th century

Gauss: late 18th

Zeilberger: 20th century

7. (10 points). What is an Egyptian fraction? Express $\frac{5}{6}$ as an Egyptian fraction

Ans. to 7: Expressing a fraction as a sum of unit fractions (pure reciprocals). $\frac{1}{2} + \frac{1}{3}$.

8. (10 points) What is the difference between Ionian (Greek) mathematics and ancient Babylonian and Chinese mathematics? Who was the traditional father of Greek mathematics?

Ans. to 8: The former was *pure* the latter was 'applied', practical, and not proof-based. Thales of Milete.

9. (10 points) What book, except for the bible, was the most reproduced and studied in the Western world? Who was its author?

Ans. to 9: 'The Elements', by Euclid.

10. (10 points) In a closed polyhedron, what is a relation between V, the number of vertices, E, the number of edges, and F, the number of faces? Who is it due to?

Ans. to 10: $V - E + F = 2$.

11. (10 points) What is the symbol, and name, of the following constant:

$$\lim_{n\to\infty} \left(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n} - \log n \right) \quad .$$

What is its approximate value?

Ans. to 11: Euler's constnant $\gamma = 0.57721\ldots$.