Dr. Z.'s Challenge Problems for Math436

Last Update: April 30, 2017.

Please submit your answers, in a separate sheet of paper, by the due date. At end of the semester, the three top scorers got a π-T-shirt (with many formulas involving π). [Originally the prize was going to be "How to Solve It" by George Polya, but it is a free download from the web].

The winners are also listed in the webpage of this class

`http://www.math.rutgers.edu/~zeilberg/math436_17.html` .

Added April 30, 2017: This ends this contest. The three top scores were:

- Zongzie Deng: 595 points
- Lauren Squillace: 397 points
- Tyler Volpe : 185 points

Also ran: Poorva Sampat (87 points), Suiliang Ma (60 points), and Cassanda Sanchez (50 points)

Rules: It is the honor system. Do not get help from anyone, or consult books or the internet. If you do, you must mention what help you got, and you may get partial credit.

Thanks to Felix Lazebnik for inspiring some of the problems here.

1. [10 points] (posed Jan. 23, 2017, solved Jan. 26 by Lauren Squillace and Poorva Sampat)

In class we realized that in order to be able to weigh any merchandise from 1 to 63 grams , where one side has the merchandise, and the other side has the fundamental weights, the most efficient way is to have fundamental weights of

$$1, 2, 4, 8, 16, 32 \quad grams$$

Suppose that you are now allowed to place the fundamental weights on **both** sides, so for example to weigh a merchandise of 7 grams instead of putting 1, 2, and 4 on one side, and the merchandise by itself on the other side, you can put 8 on the left, and the merchandise together with 1 on the right side, since $7 = 8 - 1$.

Question What is the **smallest** number of fundamental weights that would enable you to measure any merchandise from 1 to 40 grams, with the second convention? What are these fundamental weights? [The original problem had 80 rather than 40, the new version is a bit easier, and more natural]

Answer: $1, 3, 9, 27$. For example $40 = 1 + 3 + 9 + 27$, $16 = 27 - 9 - 3 + 1$, etc.

2. [10 points] (posed Jan. 24, 2017, due Jan. 30, 2017. Solved by Ben Vreeland, Lauren Squillace and Poorva Sampat)

I am a positive integer. Using the following clues, find who I am.

- If you divide me by 3 you get 2
- If you divide me by 5 you get 4
- If you divide me by 7 you get 6
- If you divide me by 11 you get 10
- If you divide me by 13 you get 12
- If you divide me by 17 you get 16
- If you divide me by 19 you get 18
- If you divide me by 23 you get 22
- I am as small as can be under the above conditions.

Answer:

$$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 - 1 = 11,546,434$$

3. [6 points] (posed Jan. 24, 2017, due Jan. 30, 2017 Solved by Lauren Squillace and Poorva Sampat)

Suppose that there are 60 people who have to equally share 77 pizzas.

A stupid way would be to cut each pizza into 60 equal slices, each one getting a very unappetizing 77 very thin pieces. Can you do it in such a way that no pizza is cut in more than five equal pieces?

Ans. The simplest Egyptian Fraction representation of $\frac{77}{60}$ is

$$\frac{77}{60} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

Taking common denominator this equals

$$\frac{30}{60} + \frac{20}{60} + \frac{15}{60} + \frac{12}{60}$$

So

- you take 30 pizzas and cut them each into two equal pieces.
- you take 20 pizzas and cut them each into three equal pieces.
- you take 15 pizzas and cut them each into four equal pieces.
- you take 12 pizzas and cut them each into five equal pieces.

4. [16 points] (posed Jan. 24, 2017, due Jan. 30, 2017. Solved by Lauren Squillace and Poorva Sampat)

Daddy told his three daughters: Alice, Barbara, and Celia, before they went to bed, that he is going to bake them cookies, and they should divide it **equally** between themselves, so that every girl gets the same number of cookies. If they are left-overs (i.e. the amount of cookies is not a multiple of 3), then the dog will get the left-over(s).

In the middle of the night, Alice woke up and saw the cookies. After she split them into three parts, there was one cookie left, that she gave to the dog. She ate her part, and went back to sleep.

One hour later, Barbara woke up. Not realizing that Alice already took her part, she did the same thing, once again, after dividing the cookies into three parts, there was one left for the dog. She ate her part, and went back to sleep.

One hour later, Celia woke up. Not realizing that Alice and Barbara already took cookies, she did the same thing, once again, after dividing the cookies into three parts, there was one left for the dog. She ate her part, and went back to sleep.

In the morning the remainig cookies could be divided equally into three parts, and the poor dog got nothing. Of course, every girl kept her night binge to herself, pretending that the remaining cookies were all that Daddy baked.

What is the smallest amount of cookies that makes this possible?.

Answer: 25 cookies.

- Alice gave the dog one cookie, and ate $(25-1)/3 = 8$ cookie, leaving $25-1-8 = 16$ cookies.

- Barbara gave the dog one cookie, and ate $(16-1)/3 = 5$ cookie, leaving $16-1-5 = 10$ cookies.

- Celia gave the dog one cookie, and ate $(10-1)/3 = 3$ cookie, leaving $10-1-3 = 6$ cookies.

In the morning, each girl got two cookies, and the dog got none.

This is the **smallest** answer. If you add any multiple of $3^3 = 27$, you also get a valid answer.

Clever way: Each operation is

$$x \to \frac{2}{3}(x-1) \quad .$$

If x is positive, this always shrink, but if $x = -2$ this is the same!, since

$$-2 \to \frac{2}{3}(-2-1) = \frac{2}{3}(-3) = -2 \quad .$$

So $x = -2$ is also a solution! (After Alice did her part, there were -2 cookies left, similarly, after Barbara, and after Celia) Alas, it is negative. Adding 3^3, we get that $-2+3^3 = 25$ is the smallest *positive* answer.

5. [15 points altogether] (posed Jan. 30, 2017, due Feb. 2, 2017. Both parts solved by Lauren Squillace, Poorva Sampat, and Tyler Volpe)

An $m \times n$ **magic rectangle** is an array with m rows, each of length n, such that each integer in $\{1, 2, 3, \ldots, mn\}$ appears exactly once, and all rows sum to the same thing, and all columns sum to the same thing (of course, the row-sums and column-sums are different if $m \neq n$).

(a) [5 points] Construct a 2×5 magic rectangle, or prove that none exists.

Ans. Impossible, since $1+2+\ldots+10 = 55$, so each row has to some up to $\frac{55}{2}$ and this is not an integer.

(b) [10 points] Construct a 2×4 magic rectangle, or prove that none exists.

Ans. There are quite a few answers, here is one

1	4	6	7
8	5	3	2

6. [10 points] (Posed Jan. 30, 2017, due Feb. 2, 2017; Solved by Cassandra Sanchez and Lauren Squillace)

Construct a "Magic word square", where the first row and first column consist of your name.

Cassandra Sanchez's answer

$$\begin{matrix} C & A & S & S \\ A & Q & U & A \\ S & U & R & F \\ S & A & F & E \end{matrix}$$

Lauren Squillace's answer

$$\begin{matrix} L & A & U & R \\ A & C & N & E \\ U & N & I & T \\ R & E & T & E \end{matrix}$$

Note: For many computer-generated magic word squares, for most of the students in this class (as well as for the prof (DORON), and for MATH) see the files in the directory:

`http://www.math.rutgers.edu/~zeilberg/math436/magic/`

7. (10 points, due Feb. 6, 2017. No computers!) [Solved by Lauren Squillace and Zongjie Deng] What are the last two (decimal) digits of $7^{100000002}$? Explain!

Sol. Since $7^4 \; mod \; 100 = 1$, $7^{100000002} \; mod \; 100 = 7^{100000000} \cdot 7^2 \; mod \; (100)$. Since 100000000 is divisible by 4, $7^{100000000} \; mod \; 100 = 1$ so the answer is $7^2 \; mod \; 100 = 49$.

Ans.: The last two digits of this (huge) number are 49.

8. (20 points, due Feb. 6, 2017. No computers) [Solved by Lauren Squillace and Zongjie Deng] What are the last two (decimal) digits of $77^{100000002}$? Explain!

Sol. $77 = 7 \cdot 11$. We already know that $7^{100000002} \; mod \; 100 = 49$.

By looking at successive powers of 11, we get that $11^{10} \; mod \; 100 = 1$, so $11^{100000002} \; mod \; 100 = 11^2 \; mod \; 100 = 21$.

Since $21 \cdot 49 = 1029$, the last two digits are 29.

Ans.: The last two digits of this (huge) number are 29.

9. (20 points, due Feb. 13, 2017. Only solved by Zongjie Deng) It is known from general theory that

$$A(n) = \sum_{i=1}^{n} i^{999999}$$

is a polynomial of n of degree 1000000. Hence it can be written as

$$A(n) = a_0 n^{1000000} + a_1 n^{999999} + \ldots + a_{9999999} n + a_{1000000}$$

(where, of course $a_{1000000} = 0$ (why?)). What are the exact values of a_0 and a_1?

Answer: $a_0 = \frac{1}{10000000}$, $a_1 = \frac{1}{2}$.

Solution: It is easier to do the first leading coefficients of $\sum_{i=1}^{n} i^k$. Writing it as $a_0 n^{k+1} + a_1 n^k + \ldots$, we have

$$(n+1)^k = a_0((n+1)^{k+1} - n^{k+1}) + a_1((n+1)^k - n^k) + \ldots$$

By the binomial theorem

$$n^k + kn^{k-1} + \ldots = a_0(n^{k+1} + (k+1)n^k + (k+1)k/2n^{k-1} + \ldots - n^{k+1}) + a_1(n^k + kn^{k-1} + \ldots - n^k) + \ldots$$

Simplifying

$$n^k + kn^{k-1} + \ldots = a_0((k+1)n^k + (k+1)k/2n^{k-1} + \ldots) + a_1(kn^{k-1}) + \ldots$$

Simplifying more

$$n^k + kn^{k-1} + \ldots = ((k+1)a_0)n^k + ((k+1)k/2a_0 + ka_1)n^{k-1} + \ldots$$

Comparing the coefficient of n^k we get

$$1 = (k+1)a_0 \quad ,$$

so $a_0 = \frac{1}{k+1}$. Comparing the coefficient of n^{k-1} we get

$$((k+1)k/2a_0 + ka_1 = k$$

So

$$ka_1 = \frac{k}{2} \quad ,$$

so $a_1 = \frac{1}{2}$.

10. (10 points, due Feb. 13, 2017, [extended from Feb. 9]. Solved by Zongjie Deng, Tyler Volpe, and Lauren Squillace)

Two trains 100 miles apart are traveling toward each other along the same track. They both go 50 miles per hour; A bird is flying just above the nose of the first train towards the second train, turns around immediately, flies back to the first train, and turns around again. It goes on flying back and forth between the two trains until they collide. If the bird's speed is 300 miles per hour, how far will it travel?

Ans.: 300 miles. (The train meet after $100/(50 - (-50)) = 1$ hour, and the bird goes 300 mph, so it traveled $1 \cdot 300 = 300$ miles.

Comment: The stupid way would have been to figure out the duration of each leg of the bird's trip, and add-up a complicated geometric series.

11. (5 points, due Feb. 13, 2017, [extended from Feb. 9]. New deadline: Feb. 23, 2017. No one got it.)

Archaeologists in Egypt found the skeletons of seven ancient cows, and using carbon dating, they estimate that they died in the time of Joseph and Pharaoh. Did they find the skeletons of the seven fat cows, or the seven skinny cows?

Ans. The cows were in Pharaoh's **dream**, hence the answer is **neither**.

12. (15 points, due Feb. 13, 2017 [extended from Feb. 9]. Solved by Zongjie Deng, Tyler Volpe, and Lauren Squillace).

The sergeant-major of an army regiment wanted to arrange his soldiers in a rectangular formation.

First he tried to arrange them in pairs, but there was one soldier left.

Next he tried to arrange them in triples, but there was one soldier left.

Next he tried to arrange them in quadruples (columns of four), but there was one soldier left.

Next he tried to arrange them in columns of five each, but there was one soldier left.

Next he tried to arrange them in columns of six each, but there was one soldier left.

Finally, he tried to arrange them in columns of seven each, and he succeeded, there was no one left.

What is the smallest number of soldiers that makes the above scenario possible?

Ans.: 301 soldiers.

13. (15 points, due Feb. 13, 2017. Solved by Zongjie Deng) It is a fact that 2^{29} has all **different** (decimal digits), and it has all the ten digits, *except* for one of them. Without computer, figure out that missing digit. Explain!

[Hint, recall the divisibility test by 9: the sum of the digits must be divisible by nine. More generally, the remainder of *any* integer (written in the usual, decimal, notaion) upon division by 9 is the same as the remainder of the division of the sum of the digits by 9 (since $10^i \, mod \, 9$ equals 1 for all $i \geq 0$).

Ans. The digit 4 is missing.

Sol.$2^6 \, mod \, 9 = 1$, so $2^{29} \, mod \, 9 = 2^{24+5} \, mod \, 9 = 2^{24} \cdot 2^5 \, mod \, 9 = 32 \, mod \, 9 = 5$. So the sum of the digits of 2^{29} mod 9 is 5. But if all the ten digits are different it is $0 + 1 + 2 \ldots + 9 = 45$ minus the missing one (mod 9). Since $45 \, mod \, 9 = 0$, the missing one is $-5 \, mod 9 = 4$.

14. (10 points, due Feb. 16, 2017. Solved by Lauren Squillace and Tyler Volpe) In a certain planet there are two kinds of tribes, The Truth-Tellers and Lie-Tellers. Truth-Tellers *always* tell the truth, and Lie-Tellers *always* lie.

You meet two creatures: Alice and Bob. Alice tells you that Bob is a Lie-Teller, and Bob says, Neither Alice nor I are Lie-Tellers".

Can you determine who is a Truth-Teller and who is a Lie-Teller?

Ans.: Alice is a Truth-Teller, Bob is a Lie-Teller

15. (20 points, due Feb. 16, 2017. Solved by Tyler Volpe)

In a certain planet there are two kinds of tribes, The Truth-Tellers and Lie-Tellers. Truth-Tellers *always* tell the truth, and Lie-Tellers *always* lie.

You meet three creatures: Celia, Heather and Dave.

Celia tells you, Dave is a Lie-Teller.

Heather says, It's false that Celia is a Lie-Teller.

Dave claims, Celia could say that I am a Lie-Teller.

Can you determine who is a Truth-Teller and who is a Lie-Teller?

Ans. Celia and Heather are Lie-Tellers, Dave is a Truth-Teller.

16. (25 points, due Feb. 16, 2017. Solved by Tyler Volpe) In a certain planet there are two kinds of tribes, The Truth-Tellers and Lie-Tellers. Truth-Tellers *always* tell the truth, and Lie-Tellers *always* lie.

You meet four creatures: Doron, Alex, Bob and Zoe.

Doron tells you that Bob could say that Alex is a Lie-Teller.

Alex claims, I would tell you that Bob is a Lie-Teller

Bob claims, Only a Lie-Teller would say that Zoe is a Lie-Teller.

Zoe claims, I and Doron are different.

Can you determine who is a Truth-Teller and who is a Lie-Teller?

Ans. Not possible.

17. (15 points, due Feb. 20, 2017. Solved by Suiliang Ma, Lauren Squillace and Zongjie Deng)

A tourist was walking through a jungle when he was made prisoner by cannibals that love logic. She was brought before the king and told,

you may now say one sentence. If the sentence is true, then we will hang you, If your statement is false, we will shoot you.

The tourist thought for a moment and then she uttered a sentence. Puzzled, the clever cannibals realized that they must let her go.

What did the tourist say to them?

Ans.: The tourist said: You will shoot me. (if they will hang her, the sentence is false, hence they have to shoot her. If they will shoot her, the sentence is true, so they will have to hang her, so their pledge can's be done, and they (reluctantly) let her go.

18. (15 points, due Feb. 20, 2017. Solved by Suiliang Ma and Zongjie Deng)

In a certain country there are only two kinds of coins, one is worth 1001 dollars each, and the other is worth 989 dollars each. You and the cashier have an unlimited supply of each type.

How would you pay for a cup of coffee that costs exactly 1 dollar (without tipping!). In other words, what kind, and how many, coins do you have to give to the cashier, and what kind, and how many, does the cashier return to you?

Ans.: You would give the cashier 417 989-dollar coins, and get back, as change, 412 1001-dollar coins. Since $417 \cdot 989 - 412 \cdot 1001 = 1$, it works out.

19. (20 points, due Feb. 23, 2017. OK to use computers; Solved by Zongjie Deng and Lauren Squillace)

Find all pairs of integers x, y, satisfying $1000 \geq x > y > 0$ such that

$$x^2 - 3y^2 = 1 \quad .$$

Ans. $(x, y) = (2, 1), (7, 4), (26, 15), (97, 56), (362, 209)$.

20. (20 points, due Feb. 23, 2017. No computers! Solved by Zongjie Deng)

Prove that there do not exist pairs of positive integers x, y, satisfying

$$x^2 - 3y^2 = -1 \quad .$$

Sol.

If $x \ mod\ 3 = 0$ then $x^2 \ mod\ 3 = 0$

If $x \ mod\ 3 = 1$ then $x^2 \ mod\ 3 = 1$

If $x \ mod\ 3 = 2$ then $x^2 \ mod\ 3 = 1$

But $3y^2 - 1$ mod $3 = 2$ so it is never a square.

21. (20 points, due Feb. 23, 2017. OK to use computers. Solved by Zongjie Deng and Lauren Squillace)

Find all pairs of integers x, y, satisfying $1000 \geq x > y > 0$ such that

$$x^2 - 7y^2 = 1$$

Ans. $(x, y) = (1, 0), (8, 3), (127, 48)$.

22. (20 points, due Feb. 23, 2017. No computers! Solved by Zongjie Deng)

Frove that there do not exist pairs of positive integers x, y, satisfying

$$x^2 - 7y^2 = -1$$

If x mod $7 = 0$ then x^2 mod $7 = 0$

If x mod $7 = \pm 1$ then x^2 mod $7 = 1$

If x mod $7 = \pm 2$ then x^2 mod $7 = 4$

If x mod $7 = \pm 3$ then x^2 mod $7 = 2$

But $7y^2 - 1$ mod $7 = 6$ so it is never a square.

23. (25 points, due Feb. 27, 2017. Solved completely by Zongjie Deng, and partially by Lauren Squillace (who gets 10 points))

A watermelon is 99% water. A cargo of one thousand kilograms of watermelons was shipped. and during the shipment, some water evaporated. The watermelons that arrived are made up of 98% water. What was the weight of the shipment when it arrived

Sol. Initially there were 10 kg of non-water. After the water evaporated, there are still 10 kg of non-water, but now it is %2 of the cargo rather than %1. Hence the current weight of the cargo is $10/(0.02) = 10/(1/50) = 500$ kg.

Ans.: The current weight of the cargo is 500 kg.

24. (25 points, due Feb. 27, 2017. Solved by Zongjie Deng)

It takes four days for a motorboat to travel from A to B down a river, and it takes five days to come back. How long will it take a wooden log to be carried from A to B by the current?

Sol. Let v be the speed of the motorboat and w the speed of the current. Let S be the distance from point A to point B. Since time= distance/velocity

$$4 = \frac{S}{v+w} \quad , \quad 5 = \frac{S}{v-w} \quad .$$

Hence

$$v+w = \frac{S}{4} \quad , \quad v-w = \frac{S}{5} \quad .$$

Subtracting

$$2w = \frac{S}{4} - \frac{S}{5} = S(\frac{1}{4} - \frac{1}{5}) = \frac{S}{20} \quad .$$

Hence

$$w = \frac{S}{40} \quad .$$

Hence the time it takes the wooden log to travel from A to B is

$$\frac{S}{w} = \frac{S}{S/40} = 40 \quad .$$

Ans. to 24: It takes a wooden log 40 days to get from A to B.

25. (25 points. Due March 2, 2017. Solved completely by Cassandra Sanchez. Solved partially by Poorva Sampat and Lauren Squillace (who get 20 points))

I overheard two people talking in a bookstore (but I could not see them, or the books that they were looking at).

A says to B: Look at that book, it is a perfect present for my three daughters, since the number that appears in the title is **exactly** the product of their ages, and all of them have different ages.

B says to A: From this information, I **know** exactly the ages of your daughters, and he told A, but he spoke so softly that you I could not hear what he said.

What is the title of the book, and what are the ages of A's three daughters?

Cassandra Sanchez's Beautiful Solution: The book's title is the product of three distinct primes (since B was able to figure out the daughter's ages, if for example the book was "around the world in eighty days", then $80 = 10 \cdot 4 \cdot 2$ but also $80 = 5 \cdot 8 \cdot 2$, so there are several options). While it is impossible to know all the books ever published, $7 \cdot 11 \cdot 13 = 1001$ works nicely, and this is a famous classic "1001 Arabian Nights".

Ans. "One thousand and one nights", often abbreviated 'Arabian Nights'.

Comment: Lauren Squillace and Poorva Sampat offered "Catch 22", and the daughters' ages as $1, 2, 11$. I would not recommend this book even to a precocious eleven-year-old, let alone to a one or two year-old!

26 (40 points altogether, due March 2, 2017; **Extended deadline for (b) March 6, 2017**. Both parts solved (see below)) **Corrected Feb. 28, 2017, thanks to Zongjie Deng**

a. (10 points) [Solved by Poorva Sampat and Zongjie Deng] Prove or disprove that there exists a number of the form $1111\ldots111$ that is divisible by 12345.

Solution: Any multiple of 12345 is a multiple of 5, hence must end with 5 or 0, and $111\ldots111$ never does.

b. (Was 30 points, now 40 points; **Deadline extended to March 6, 2017.** Solved by Zongjie Deng) Prove or disprove that there exists a number of the form $1111\ldots111$ that is divisible by 12347.

Solution 1 (Zongjie Deng's nice (but too advanced) solution).

By Euler's identity $10^{\phi(12347)}\ mod\ 12347 = 1$. Hence $10^{\phi(12347)} - 1$ is divisible by 12347, and hence (since $gcd(9, 12347) = 1$), $(10^{\phi(12347)} - 1)/(10 - 1)$ is. But since $\sum_{i=0}^{n-1} 10^i = (10^n - 1)/(10 - 1) = (10^n - 1)/9$ we get that $11111\ldots1$ with 1 repeated $\phi(12347) - 1$ times, is divisible by 12347.

Solution 2 (elementary!).

Consider the sequence $a_n := 1\ldots1$(1 repeated n times) modulo 12347, for $1 \le n \le 12348$. There are 12347 possible values, but the sequence has 12348 entries. They can't be all different! (the pigeon-hole principle). Hence there are $1 \le n_1 < n_2 \le 12348$ such that $a_{n_1} = a_{n_2}$. Hence $1\ldots1$ (repeated n_2 times) minus $1\ldots1$ (repeated n_1 times) is divisible by 12347. But $1\ldots1$ (repeated n_2 times) minus $1\ldots1$ (repeated n_1 times) equals $1\ldots1$ (repeated $n_2 - n_1$ times) times 10^{n_1}. Since $gcd(10, 12347)$, it follows that $1\ldots1$ (repeated $n_2 - n_1$ times) is divisible by 12347.

27 (20 points, due March 2, 2017; **Deadline extended to March 23, 2017** Solved by Tyler Volpe.)

There are six people, every two people either love each other or hate each other (they have strong feelings, no one is indifferent to any person. Also there is all the feelings are mutual).

Prove that there are either three people who all love each other, or three people who all hate each other (or both kinds).

Hint added March 6, 2017: Consider one of the people, let's call her Alice. Either Alice has more lovers than haters, or more haters than lovers. In other words, she either has at least three lovers or at least three haters.

Tyler Volpe's Solution Alice either loves at least three people or hates three people. Suppose the former. Pick any three people Alice loves (If she only loves three people pick them all, if she loves four or five people, pick any three of them). Then among these three Alice-lovers, either they all hate other, and we found a hate-triangle, or at least two of them love each other, and together with Alice they form a love triangle. If Alice hates more people than she loves, reverse love and hate in the previous argument.

Comment: This is a special case of a famous theorem in cominatorics that says that if you have $(2n)!/n!^2$ people, then you are guaranteed either n people who all love each other or n people who all hate each other. It is called Ramsey's theorem.

28 (25 points. Due March 6, 2017. Solved by Zongjie Deng and Lauren Squillace)

In a certain very small Mathematics PhD program, there are only five professors, and five PhD students. Each professor has exactly one graduate student. The professors are very jealous, and get very angry if they find out that their student is working on another professor's problems.

Because of gossip, *everyone* knows about the other students, but of course, not about his (or her) own student.

One day, the dean (not one of the five professors) tells them:

At least one of the students is working on problems that are not their professor's. If a professor knows for *sure* that his or her student is working with someone else, he should kick him or her out of the PhD program. This has to be done on 4pm every day.

The first day, nothing happened, the second day, nothing happened, the third day, nothing happened, the fourth day, nothing happened, but at 4pm of the fifth day, all five professors kicked their students out, since each realized that their student was working with someone else.

Why did it take them so long? How did they find out? Explain!

Solution: Since every professor kicked his student. It turned out that all the students were cheating. Why did it take them five days to realize? Of course, every one knows about the other professors and students, but everyone assumes that *his* student is innocent. This is the *default assumption.*

Suppose that there is only one professor and one student. Then after the dean's announcement, the one professor knows for sure that his student is cheating, and does kick him out after **one** day.

Suppose that there are only two professors and two students. Then under the default assumption, each professor expects his colleague to kick his student out after the first day. Since that does not happen, they both realize that their default assumption was wrong, and after two days kick the students out.

Suppose that there are only three professors and three students. Then under the default assumption, this is the same as two professors and two students. By the above paragraph, each professor expects his two colleagues to kick their students out after two days. Since that does not happen, the three of then realize that their default assumption was wrong, and after three days, they kick the students out.

Theorem: With n professors and n studends, and the above story, all of them cheating, it takes n days for them to realize it.

Proof: For $n = 1$ this is obvious. By the default assumption, the other $n-1$ professors would have realized the cheating after $n-1$ days. Since they did not, the default assumption is wrong, and everyone realizes that their student is cheating on them. Hence they all fire their students after n days.

Now take $n = 5$.

29 (15 points. Deadline extended. Currently due March 30, 2017. Closed. No one got it)

Prove that there are at least two Rutgers students with *exactly* the same number of hairs.

Sol. The maximum number of hairs that a person can have is 30000, there are 50000 Rutgers students, they can't all have a different number of hairs.

30 (15 points, Due March 9, 2017) [OK to use computers] Solved by Zongjie Deng (by hand!).

Evaluate the definite integral

$$\int_0^1 \frac{x^4\,(1-x)^4}{1+x^2}\,dx \quad .$$

Ans. $\frac{22}{7} - \pi$.

31 (25 points; New Deadline March 23, 2017. Completely solved by Zongjie Deng, partially solved (numerically) by Tyler Volpe, who gets 12 points).

Let a_n be the probability that if you toss a fair coin $2n$ times, you get **exactly** n Heads and **exactly** n tails. Find the limit

$$\lim_{n\to\infty} \frac{1}{n a_n^2} \quad .$$

Hint: $a_n = ((2n)!/n!^2)/2^{2n}$. You can use a computer to estimate the limit by plugging-in large values of n.

Ans.: π. To prove it, use Stirling's formula that $n!$ is asympototics to $\sqrt{2\pi n}(n/e)^n$.

32 (15 points, due March 9, 2017; Closed)

What is special about the following sentence

"How I need a drink, alcoholic of course, after the heavy chapters involving quantum mechanics"?

Can you devise a longer sentence with the same intention?

Ans.: The word-lengths are the beginning digits of π.

33 (15 points; Due March 23, 2017; Deadline extended to March 27, 2017.) Make up a nice stentence (or paragraph) for the digits of

$$e = 2.71828182845904523536028747135266249775724 7\ldots \quad ,$$

where the word length correspond to the digits.

34 (15 points; Due March 23, 2017. Solved by Cassandra Sanchez) Make up a nice stentence (or paragraph) for the digits of

$$\phi = 1.6180339887498948482045868343656381177203\ldots \quad ,$$

where the word length correspond to the digits.

Cassandra Sanchez's solution:

I, Cassie, a frazzled - cat can autograph jejunely puzzling budgets.

35. (20 points, due March 27, 2017. Still open, deadline extended to March 30, 2017. Solved by Tyler Volpe)

Alice and Bob are good at logic, and I put two consecutive positive integers on their foreheads. Alice can see Bob's number but not her own (there are no mirrors) and similary, Bob can see Alice's number but not his own.

In Round 1, I ask: Does any of you know for sure your own number? No one did.

In Round 2, I ask: Does any of you know for sure your own number? No one did.

and so on, in the first 19 rounds no one knew. Finally,

In Round 20, I ask: Does any of you know for sure your own number? and Alice said, I do?

What was her number and what was Bob's number. What took her so long?

Hint (written March 27, 2017): As warm-up, replace 20 by 1, then by 2, then by 3, and see what is going on.

Sol. If Alice had the number 1 on her head, Bob would now right away that his number is 2 (since it can't be 0), and would say so in the first round. Ditto for Bob. Since no one said anything the first round, they **both** know (being smart) that the smallest number that can show up is 2. Of one of them would have seen 2 at the second round they would say so. If no one said anything, they both know that the smallest number that can show up is 3. You keep going until Round 20.

36. (15 points, due March 27, 2017. Solved by Zongjie Deng) In a raquetteball court, a ball hits the left-wall, then the front wall, then the ceiling. Prove that the ball's direction is exactly opposite to its original direction.

Sol. If the initial direction vector of the ball was (x, y, z) then after hitting the yz-wall it is $(-x, y, z)$, after hitting the xz-wall it is $(-x, -y, z)$, after hitting the xy-wall it is $(-x, -y, -z)$ which is opposite.

A more clever solution: Put mirrors on all the walls and the ceiling and the floor, then the mirror-image of the ball's trajectory is a straight line, and after hitting all the walls and the floor and the ceiling, it is back to its original direction, so half-way it is at the opposite direction.

37. (25 points, due March 27, 2017. Solved by Zongjie Deng) In an n-dimensional raquetteball court, a ball hits all n faces (in any order) all adjacent to one corner Prove that the ball's direction is exactly opposite to its original direction.

Sol. Similar to the above, but with vectors of length n.

38. (30 points, due March 27, 2017; Solved by Lauren Squillace)

Ten bad guys went to Hell, and in order to atone for their sins, and get transfered to Heaven, they have to do ten different unpleasant chores, multiple times. After they finish it they have to follow instructions

If today you are in Station 1, go tomorrow to Station 5

If today you are in Station 2, go tomorrow to Station 4

If today you are in Station 3, go tomorrow to Station 6

If today you are in Station 4, go tomorrow to Station 7

If today you are in Station 5, go tomorrow to Station 1

If today you are in Station 6, go tomorrow to Station 8

If today you are in Station 7, go tomorrow to Station 2

If today you are in Station 8, go tomorrow to Station 9

If today you are in Station 9, go tomorrow to Station 10

If today you are in Station 10, go tomorrow to Station 3

They are done when all of them returned to their original positions. How long did it take them to go to Heaven?

Sol. This is a permutation of length 10

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 4 & 6 & 7 & 1 & 8 & 2 & 9 & 10^3 & \end{pmatrix}$$

Breaking it into cycles (in cycle notation) this is

$$(1,3)(2,4,7)(3,6,8,9,10) \quad .$$

The cycle-lengths are $2, 3, 5$ so the smallest i such that π^i is the identity permutation is $lcm(2,3,5) = 30$.

Ans.: 30 days.

39. (30 points, due, March 30, 2017. Closed, solution presented in class) Find a way, if possible, to cover a 12×12 board where the top-left and bottom-right squares have been removed

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by $(12^2 - 2)/2 = 71$ domino-pieces (i.e. 1×2 or 2×1 rectangles), or else, prove that it is impossible.

Ans. If you color the 12×12 board like a Checker-Board, by Black and White, the two removed opposite-corner squares have the **same color**. Every time you place a domino piece, you cover **exactly** one white and **exactly** one black square. Hence in any legal covering there are as many black as white squares, but the above board does not have the same number of black and white squares.

40 (30 points, due, March 30, 2017; Only solved by Suiliang Ma)

An absent-minded professor has to send n recommendation letters so she prepared them ahead of time, and she also prepared n enevelopes. She is so absent-minded that she puts each letter to a random envelope (but only one letter in each), in how many ways can she do it so that **no** letter was sent to the right address?

Sol.

$$n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \ldots + \binom{n}{n} 0!$$

(See wikipedia article on derangements)

41. (30 points, due April 3, 2017. Solved by Zongjie Deng)

Using the fact that the set of non-zero integers modulo a prime, p, $\{1, 2, \ldots p-1\}$ is a group, and that for any a, the set of its powers is a subgroup, use Largrange's theorem to prove Fermat's Little Theorem that $a^{p-1} \; modp = 1$.

Sol. The order of an element in a group G is the smallest integer i such that $a^i = e$, where e is the identity. The set of elements $\{e, a, a^2, \ldots, a^{i-1}\}$ is a subgroup (why?). It has i elements, hence the number of elements in the group divided by the order (of any element) is an integer. The number of elements in the group whose elements are $\{1, \ldots, p-1\}$ and where the operation is multplication modulo p, is $p-1$, hence $(p-1) = ik$ for some integer k, hence $a^{p-1} = a^{ik} = (a^i)^k = 1^k = 1$ modulo p.

42. (20 points, due April 3, 2017. Solved by Lauren Squillace and Zongjie Deng)

A man is looking at a photograph of someone. His friend asks who it is. The man replies, 'Brothers and sisters, I have none. But that man's father is my father's son'. Who is in the photograph?

Ans.: The man's son.

43. (20 points, due April 6, 2017. Solved by Lauren Squillace, partially using graph theory, she gets 10 points)

A farmer has to cross a river with a sheep, a wolf, and a huge cabbage. He has a row-boat that can only have him and one of the above. At no time can the sheep and Cabbage left alone, and at no time can the sheep and the wolf be left alone (but the Cabbage is safe with the wolf alone).

Using a graph (a graph-theory graph, not a calculus one!) (no credit for other ways!) find a way for the farmer to do the multiple crossings, bringing every one accross the river safely.

Sketch of the solution: You start out with $FCSW$ on one bank, and the empty set on the other bank. A boat-load can have either F by himself, or FC, FS, or FW. You draw a graph whose vertices are legal subsets of $FCSW$ keeping in mind that,e.g., SW is no good and CS is no good, but also FC is no good, since on the other bank you would SW. You draw a directed edge, if you can go from one legal position on the left bank another one by one of the legal boat loads. You then look for an **alternating** path with the arrow, followed with 'against the arrow', corresponding to travelling back and forth. You look for a such an alternating path in the directed graph starting with $FCSW$ and ending with the empty set.

44. (35 points, due April 6, 2017. Solved by Lauren Squillace, partially using graph theory, she gets 25 points)) Three cops went to the jungle and caught three run-away-prisoners. They have to cross a river with a row-boat that can only have one or two persons at at time. At no time (on either side of the river) should there be more prisoners than cops, or else they would murder the cops and escape.

Using a graph (a graph-theory graph, not a calculus one!) (no credit for other ways!) find a way for the cops to arrange the crossings, back-and-forth, so that the prisoners and cops will go to the other side. All the cops and all the prisoners know how to row a boat, and the prisoners, also want to go the other side, or else they would starve.

Sketch of the solution: Similar to the above, where the positions are (i, j) indicating i cops and j prisoners, where $i \geq j$ if $j > 0$ and $3 - i \geq 3 - j$ if $j < 3$.

45. (40 points, due April 10, 2017. Solved by Lauren Squillace and Tyler Volpe) The Beatles have a concert that starts in 34 minutes and they must all cross a rope bridge to get there. All four men begin on the same side of the bridge. You must help them across to the other side. It is night. There is one flashlight. A maximum of two people can cross at one time. Any party who crosses, either 1 or 2 people, must have the flashlight with them. The flashlight must be walked back and forth, it cannot be thrown, etc. Each band member walks at a different speed. A pair must walk together at the rate of the slower man's pace. The rates are:

- John - 2 minute to cross,
- Ringo - 4 minutes to cross,
- Paul- 10 minutes to cross,
- George - 20 minutes to cross.

Sol. (J=John, G=George, P=Paul, R=Ringo)

JRPG — 0 min

PG — JR 4 min

RPG — J 8 min

JR — PG 30 min

— JRPG 4 min

46. (40 points, due April 13, 2017. Solved (in a complicated but elegant and **correct** way) by Zongjie Deng!)

In the Passover *Seder* there is a song called "Dayenu" that lists the minimum requirement for us to be happy with God.

There are n elementary events God did but did not have to do, $A_1, A_2, \ldots, A_n$.

It 'would have been enough' (*dayenu*) if God did the following compound event

$$f(A_1, A_2, ..., A_n) = A_1\, AND\, \bar{A}_2 \quad OR \quad A_2 AND \bar{A}_3 \quad OR \quad \ldots \quad OR A_{n-1}\, AND\, \bar{A}_n \quad ,$$

where $\bar{A}$ means 'NOT A', and OR is "Inclusive OR".

If the Probability of each A_i is 1/2, and there are **independent**, what is the probability of the event $f(A_1, ..., A_n)$, i.e. the event that God passes the minimum requirement for us to be happy?

Zongjie Deng's Answer:

$$\sum_{k=1}^{n} \binom{n-k}{k} (-1)^{k+1} \frac{1}{4^k}$$

Mr. Deng's nice approach: Inclusion Exclusion and other clever tricks.

Dr. Z.'s Answer:

$$1 - \frac{n+1}{2^n} \quad .$$

Dr. Z's approach: See his latest paper

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/dayenu.pdf`

Corollary: Since both answers are correct, we have the following **Deng-Zeilberger identity**

$$\sum_{k=1}^{n} \binom{n-k}{k} (-1)^{k+1} \frac{1}{4^k} \quad = 1 - \frac{n+1}{2^n} \quad .$$

47. (30 points, due April 17, 2017. OK to use a computer (or calculator). Only solved by Lauren Squillace)

Which of the following gambles is the most favorable

- Rolling 6 (fair) dice and getting at least one six?
- Rolling 12 (fair) dice and getting at least two sixes?
- Rolling 18 (fair) dice and getting at least three sixes?
- Rolling 24 (fair) dice and getting at least four sixes?

Ans. The probability of success are, respectively

$$1 - \binom{6}{0} (\frac{5}{6})^6 = 0.665\ldots$$

$$1 - \binom{12}{0} (\frac{5}{6})^0 (\frac{5}{6})^{12} - \binom{12}{1} (\frac{5}{6})^1 (\frac{5}{6})^{11} = 0.619\ldots$$

$$1-\binom{18}{0}(\frac{5}{6})^0(\frac{5}{6})^{18}-\binom{18}{1}(\frac{5}{6})^1(\frac{5}{6})^{17}-\binom{18}{2}(\frac{5}{6})^2(\frac{5}{6})^{16}=0.5973\ldots$$

$$1-\binom{24}{0}(\frac{5}{6})^0(\frac{5}{6})^{24}-\binom{24}{1}(\frac{5}{6})^1(\frac{5}{6})^{23}-\binom{24}{2}(\frac{5}{6})^2(\frac{5}{6})^{22}-\binom{24}{3}(\frac{5}{6})^3(\frac{5}{6})^{21}=0.584\ldots$$

The probabilities of success are getting worse and worse, so the best is the first one. For more details on this historically important problem see Dr. Z.'s masterpiece

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/pepys.pdf`

[and the reference there].

48. (30 points, due April 17, 2017. Only solved by ZongJie Deng)

Prove the binomial coefficient identity

$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$$

Solution: There are n boys and n girls and n candies. You have to choose a set of n children to give the candies to (at most one candy per child). There are $\binom{2n}{n}$ ways of doing it. Another way of doing it is to first decide how many girls, let's call it k, will get a candy, k can be anything between 0 and n. Once you have decided that k girls will get a candy, you know that $n-k$ boys will get a candy. For each such k there are $\binom{n}{k}\binom{n}{n-k}=\binom{n}{k}^2$ ways of doing it, and adding up from $k=0$ to $k=n$ yields it.

49. (due April 17, 2017. Only solved by ZongJie Deng)

(a) (10 points) Conjecture what is the following equal to

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}$$

Ans. F_{n+1}

(b) (30 points) Prove it!

[By Induction]

50. (30 points, due April 17, 2017. OK to use Maple [but you don't have to] Only solved by ZongJie Deng).

Give a direct proof of the **Deng-Zeilberger identity**

$$\sum_{k=1}^{n}\binom{n-k}{k}(-1)^{k+1}\frac{1}{4^k}\quad=1-\frac{n+1}{2^n}\quad.$$

(Of course, there is already a proof, Mr. Deng got the right answer to some probability question, and Dr. Z. got a different looking answer. Since they are both correct, they must be equal. You should give a more direct proof).

Mr. Deng's proof approach: By induction (similar to the above, but more complicated).

Dr. Z.'s answer (using Maple)

Type

```
sum((-1/4)**(k+1)*binomial(n-k,k),k=1..n);
```

51. (20 points, Due April 20, 2017 [no peeking in the internet (or books)])

A fixed point of a mapping $x \to f(x)$ is a number x_0 such that $f(x_0) = x_0$. It is **stable** if $|f'(x_0)| < 1$.

Recall that the only fixed points of $x \to rx(1-x)$ are $x = 0$ and $x = (r-1)/r$.

(i) For what values of r is $x = 0$ a stable fixed point?

(ii) For what values of r is $x = (r-1)/r$ a stable fixed point?

The cut-off value of r (when it starts being unstable is called r_1).

52. (30 points, Due April 20, 2017 [no peeking in the internet (or books)])

For $r > r_1$, there are no more stable fixed points, but $x \to f(f(x))$, i.e.

$$x \to r\,(rx(1-x))\,(1 - rx(1-x))$$

has, for a while, two stable fixed points (it also has $x = 0$ and $x = (r-1)/r$ as fixed points but they are not stable).

Find the exact values of these stable fixed points of $x \to f(f(x))$, and the cut-off value of r when they stop being stable. In other words, find the exact value of r_2.