A RECURSIVE FORMULATION OF SYLVESTER'S BIJECTION BETWEEN ODD AND DISTINCT PARTITIONS.

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(ca. 1984)

A partition of an integer n is a non -increasing sequence of positive integers that sum up to n. Let DIS(n) be the set of partitions of n whose parts are all different than each other and let ODD(n) be the set of partitions of n whose parts are all odd.

It is a well known result of Euler that for every n DIS(n) = ODD(n) and there is a well known bijective proof of this fact due to Glashier. Not as well known is Sylvester's([1], Act III) bijective proof, although it is very elegant and proves much more. Indeed, let

DIS(n,k)= the set of distinct partitions of n having k subsequences of consecutive integers(e.g. 98 654 21 belongs to DIS(35,3)).

ODD(n,k)=the set of odd partitions of n with k different parts.

Sylvester's bijection establishes the fact that DIS(n,k) and ODD(n,k) are equinumerous for every n and k.

The reason why Sylvester's elegant bijection is not as well known as it deserves to be is that although the ODD(n)---> DIS(n) direction is fairly simple and has a nice graphical presentation, the other direction is rather complicated. It also takes some effort to show that the mapping in question does indeed map ODD(n,k) onto DIS(n,k). I believe that the following recursive formulation of this nice algorithm presents it in its full simplicity and makes its verification very easy.

Definition of T:ODD(n)--->DIS(n)

Write your odd partition in the format $(2a_1+1,..,2a_r+1,1^m)$

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1.[Initialize] $T(1^m)=m$

2.[make recursive call] Let $(d_3,\ldots,d_s)=T(2a_2-1,\ldots,2a_r-1)$
[the # of parts of $(2a_2-1,\ldots,2a_r-1)$ is $r-1$ its "a_1" is a_2-1]

3.[finalize] Let $d_1=a_1+r+m$ [note that $d_1=a_1$ +# of parts]

$d_2=a_1+r-1$, then

$T(2a_1+1,\ldots,2a_r+1,1^m)=(d_1,d_2,d_3,\ldots,d_s)$.

<u>Definition</u> <u>of</u> <u>S:DIS(n)--->ODD(n)</u>

Write your distinct partition as $(d_1,\ldots,d_s)$

1.[initialize] $S(m)=1^m$

2.[make a recursive call] Let $(2a_2-1,\ldots,2a_r-1)=S(d_3,\ldots,d_s)$
[from here we can determine r]

3.[finalize] Let $a_1=d_2-r+1$

$m=d_1-d_2-1$, then

$S(d_1,\ldots,d_s)=(2a_1+1,2a_2+1,\ldots,1^m)$

Using induction it is now a routine matter to verify the following facts:

(i)Both T and S are well defined

(ii)TS=I and ST=I

(iii)T maps ODD(n,k) onto DIS(n,k).

Refererence.

1.J.J.Sylvester,A constructive theory of Partitions,arranged in three acts, an interact and and exodion,American Journal of Mathemaatics 5(1882) pp.251-330.