A RECURSIVE FORMULATION OF SYLVESTER'S BIJECTION
BETWEEN ODD AND DISTINCT PARTITIONS.

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A partition of an integer \( n \) is a non-increasing sequence of positive integers that sum up to \( n \). Let \( \text{DIS}(n) \) be the set of partitions of \( n \) whose parts are all different than each other and let \( \text{ODD}(n) \) be the set of partitions of \( n \) whose parts are all odd.

It is a well known result of Euler that for every \( n \) \( \text{DIS}(n) = \text{ODD}(n) \) and there is a well known bijective proof of this fact due to Glaisher. Not as well known is Sylvester's (19 Act III) bijective proof, although it is very elegant and proves much more. Indeed, let \( \text{DIS}(n,k) \) the set of distinct partitions of \( n \) having \( k \) subsequences of consecutive integers (e.g. 98 654 21 belongs to \( \text{DIS}(35,3) \)).

\( \text{ODD}(n,k) \) the set of odd partitions of \( n \) with \( k \) different parts.

Sylvester's bijection establishes the fact that \( \text{DIS}(n,k) \) and \( \text{ODD}(n,k) \) are equinumerous for every \( n \) and \( k \).

The reason why Sylvester's elegant bijection is not as well known as it deserves to be is that although the \( \text{ODD}(n) \rightarrow \text{DIS}(n) \) direction is fairly simple and has a nice graphical presentation, the other direction is rather complicated. It also takes some effort to show that the mapping in question does indeed map \( \text{ODD}(n,k) \) onto \( \text{DIS}(n,k) \). I believe that the following recursive formulation of this nice algorithm presents it in its full simplicity and makes its verification very easy.

**Definition of** \( l: \text{ODD}(n) \rightarrow \text{DIS}(n) \)

Write your odd partition in the format \( 2a +1, \ldots, 2a +1, 1 \)
Write your odd partition in the format \((2a+1, \ldots, 2a+1, 1)^m\)

1. [Initialize] \(T(1)^m = \) m

2. [make recursive call] Let \((d_1, \ldots, d_s)^r = T(2a-1, \ldots, 2a-1)^2\)
   (the # of parts of \((2a-1, \ldots, 2a-1)\) is \(r-1\) its "a" is \(a-1\))

3. [finalize] Let \(d = a + r+1\) [note that \(d = a + \#\) of parts]
   \(d = a + r-1\), then
   \[T(2a+1, \ldots, 2a+1, 1)^m = (d_1, d_2, d_3, \ldots, d_s)^r\]

Definition of \(S: \text{DIS}(n) \rightarrow \text{ODD}(n)\)

Write your distinct partition as \((d_1, \ldots, d_s)^m\)

1. [Initialize] \(S(m) = 1\)

2. [make a recursive call] Let \((2a-1, \ldots, 2a-1)^r = S(d_1, \ldots, d_s)^m\)
   [from here we can determine \(r\)]

3. [finalize] Let \(a = d - r + 1\)
   \(m = d - d - 1\), then
   \(S(d_1, \ldots, d_s)^m = (2a+1, 2a+1, \ldots, 1)^r\)

Using induction it is now a routine matter to verify the following facts:

(i) Both \(T\) and \(S\) are well defined
(ii) \(TS = I\) and \(ST = I\)
(iii) \(T\) maps \(\text{ODD}(n,k)\) onto \(\text{DIS}(n,k)\).

Reference.