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# A SHORT ROGERS–RAMANUJAN BIJECTION

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Let $A(n)=\{(\lambda(1),\ldots,\lambda(t));\ t\geqslant 0,\ \lambda(1)+\cdots+\lambda(t)=n,\ \lambda(i)-\lambda(i+1)\geqslant 2$ for $i=1,\ldots,t-1$ and $\lambda(t)>0\}$, $C(n)=\{(\lambda(1),\ldots,\lambda(t));\ t\geqslant 0,\ \lambda(1)+\cdots+\lambda(t)=n,$ $\lambda(i)-\lambda(i+1)\geqslant 0$ and $\lambda(i)\equiv 1,\ 4 \pmod 5$ for $i=1,\ldots,t\}$. The first Rogers–Ramanujan identity [1, p. 109] states that for every $n$, the finite sets $A(n)$ and $C(n)$ have the same number of elements. Recently Garsia and Milne [2, 3] proved this result by presenting a bijection between $A(n)$ and $C(n)$. We are going to give another bijective proof which is very similar in form to that of Garsia and Milne in that it involves an iteration of two involutions, one of which ($\psi$) is equivalent to the Jacobi Triple Product Identity [1, p. 21]. However, our second involution ($\varphi$) is considerably simpler than that provided by Garsia and Milne.

$A(n)$ is trivially 'isomorphic' to

$$B(n)=\{(\lambda(1),\ldots,\lambda(t));\ t\geqslant 0,\ \lambda(1)+\cdots+\lambda(t)=n \text{ and } \lambda(1)\geqslant\cdots\geqslant\lambda(t)\geqslant t\}$$

by

$$(\lambda(1),\ldots,\lambda(t))\leftrightarrow(\lambda(1)-t+1,\ldots,\lambda(i)+2i-1-t,\ldots,\lambda(t)+t-1).$$

Let

$$X(n)=\{(j;\lambda(1),\ldots,\lambda(t));\ -\infty<j<\infty,\ t\geqslant 0,\ \lambda(1)\geqslant\cdots\geqslant\lambda(t)>0 \text{ and } \tfrac{1}{2}(5j^2-j)+\lambda(1)+\cdots+\lambda(t)=n\}$$

and let $X_e(n)$ and $X_o(n)$ denote the subsets of $X(n)$ with $j$ even and odd respectively. We are going to define an involution $\varphi$ in $X(n)-O\times B(n)$ and an involution $\psi$ in $X(n)-O\times C(n)$ which both change the parity of $j$. Thus $\varphi(X_o)=X_e-O\times B(n)$ and $\psi(X_o)=X_e-O\times C(n)$ and thus $|B(n)|=|C(n)|$. An explicit bijection $\pi: O\times C(n)\to O\times B(n)$ is given by $\pi((\lambda))=$ the last well-defined element in $((\psi\varphi)^k(\lambda))_{k=0}^{\infty}$; $\pi$ is well-defined since $X_e(n)$ is a finite set and $(\psi\varphi)^r(\lambda)=(\psi\varphi)^s(\lambda)$, $r>s\Rightarrow(\lambda)=(\psi\varphi)^{r-s}(\lambda)\Rightarrow(\lambda)$ is in the range of $\psi\Rightarrow(\lambda)\notin O\times C(n)$, a contradiction. Note that $\pi$ is an iteration of the simple mapping $\psi\varphi$ which is repeated until arrival at $O\times B(n)$. It is unlikely that a non-iterative bijection between $B(n)$ and $C(n)$ will ever be found since these sets are so different.

*Definition of* $\varphi$. Consider $(j;\lambda(1),\ldots,\lambda(t))\in X(n)-O\times B(n)$. Let $a=\mathrm{Min}\{x; 2j+x-\lambda(x)>0\}$, $\mu=2j+a-\lambda(a)$, $t^*=t+\chi(\lambda(a)=0)+\chi(\lambda(\mu)=0)$.

*Case* I: $\lambda(a)+\lambda(\mu)-5j\geq t^*$. Delete the parts $\lambda(a)$ and $\lambda(\mu)$; add 1 to each of the remaining parts and insert $\lambda(a)+\lambda(\mu)-5j-t^*$ new parts of 1; $j\leftarrow j+1$.

*Case* II: $\lambda(a)+\lambda(\mu)-5j<t^*$. Let

$$m=\mathrm{Max}\{z\leq a+2j-2;\ z+\lambda(a+2j-1-z)\leq t+5j-2\}.$$

Subtract 1 from each of the $t$ parts and create new parts $m$ and $t+5j-3-m$; $j\leftarrow j-1$. It is readily seen that $\varphi$ is an involution on $X(n)-O\times B(n)$ which changes the parity of $j$. Indeed if $\bar{a}$, $\bar{j}$, etc. are the new values of $a$, $j$ etc. after an application of Case I, then

$$\bar{j}=j+1,\qquad \bar{a}=a-1,\qquad \overline{\lambda(\bar{a})}=\lambda(a+1)+1,$$

$$\bar{\mu}=2\bar{j}+\bar{a}-\overline{\lambda(\bar{a})}=2j+a-\lambda(a+1)\geq\mu,\qquad \bar{t}=\lambda(a)+\lambda(\mu)-(5j+2),$$

so $\overline{\lambda(\bar{a})}+\overline{\lambda(\bar{\mu})}-5\bar{j}<\bar{t}$ and we arrive at Case II. Furthermore, applying Case II now yields $m=\lambda(a)$, $\bar{t}+5\bar{j}-3-m=\lambda(\mu)$ and we are back where we started. Similarly, after applying Case II we are in Case I, application of which reproduces the original element of $X(n)$.

*Definition of* $\psi$. Consider $(j;\lambda(1),\ldots,\lambda(t))\in X(n)-O\times C(n)$. Let $t_0$ be the number of parts which are $\equiv 0 \pmod 5$ and for $i=0,2,3$ let $m_i$ be the largest part which is $\equiv i \pmod 5$ (if there are no such parts $m_i=0$).

*Case* I: $t_0+j>0$ or $m_2>0$.

*Case* I(a): $5t_0+5j-3\geq m_2$. Subtract 5 from each of the $t_0$ parts which are $\equiv 0 \pmod 5$ and insert a new part of $5t_0+5j-3$, $j\leftarrow j-1$.

*Case* I(b): $5t_0+5j-3<m_2$. Delete the part $m_2$, add 5 to each of the $t_0$ parts which are $\equiv 0 \pmod 5$ and create $\frac{1}{5}(m_2-5j-2-5t_0)$ new parts of 5 each, $j\leftarrow j+1$.

*Case* II: $t_0+j\leq 0$ and $m_2=0$.

*Case* II(a): $m_3+5j-3\geq m_0$. Add $5j-3$ to $m_3$ (and sort); $j\leftarrow j-1$.

*Case* II(b): $m_3+5j-3<m_0$. Subtract $5j+2$ from $m_0$ (and sort); $j\leftarrow j+1$.

It is readily seen that $\psi$ is an involution on $X(n)-O\times C(n)$ which changes the parity of $j$.

*Examples*

(1) $\varphi(2; 11, 9, 8, 7, 6, 1, 1, 1) = (3; 10, 9, 7, 2, 2, 2)$. Here $a = 4$, $\mu = 1$, Case I applies.

(2) $\psi(-2; 15, 14, 9, 9, 8, 8, 8, 5, 5, 4, 3, 2, 1) = (-3; 14, 10, 9, 9, 8, 8, 8, 4, 3, 2, 2, 1)$. Here $t_0 = 3$, $m_2 = 2$, Case I(a) applies.

(3) $\pi((4, 1)) = (3, 2)$ since

$$(0; 4, 1) \xrightarrow{\varphi} (1; 1, 1, 1) \xrightarrow{\psi} (0; 2, 1, 1, 1) \xrightarrow{\varphi} (-1; 1, 1) \xrightarrow{\psi}$$

$$(0; 3, 1, 1) \xrightarrow{\varphi} (1; 2, 1) \xrightarrow{\psi} (0; 2, 2, 1) \xrightarrow{\varphi} (1; 3) \xrightarrow{\psi} (0; 3, 2).$$

## References

[1] G.E. Andrews, The Theory of Partitions (Addison-Wesley, Reading, MA, 1976).
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