JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **74**, 192–199 (1980)

A Lattice Walk Approach to the "inv" and "maj" q-Counting of Multiset Permutations

DORON ZEILBERGER*

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332

Submitted by G.-C. Rota

A natural interpretation of "maj" and "inv" q-counting of multiset permutations in terms of walks on a lattice with multilane highways is presented. This is applied to give a short combinatorial proof of two theorems of MacMahon and to rederive a recent result of Gessel.

INTRODUCTION

Let $\{1, 2,..., n\}$ be a fixed alphabet. To every word (alias multiset permutation) $\sigma = \sigma_1 \cdots \sigma_l$ we set (Andrews [1], Garsia [4])

$$d(\sigma) = \sum_{i=1}^{l} \chi(\sigma_i > \sigma_{i+1}),$$

$$\mathrm{maj}(\sigma) = \sum_{i=1}^{l-1} i\chi(\sigma_i > \sigma_{i+1}),$$

$$i(\sigma) = \sum_{i=1}^{l-1} \sum_{j=i+1}^{l} \chi(\sigma_i > \sigma_j)$$

(for a statement A, $\chi(A) = 1$ if A is true, $\chi(A) = 0$, if A is false).

The functions $d(\sigma)$, $\mathrm{maj}(\sigma)$, $i(\sigma)$ give respectively the number of *descents*, the *major index*, and the *number of inversions*.

Consider the n-dimensional positive lattice N^n, where $N = \{0, 1, 2,...\}$, the set of nonnegative integers. Denoting a typical point by $(m_1,..., m_n)$, we let $\mathbf{e}_j$ be the unit vector in the m_j coordinate:

$$\mathbf{e}_j = (0, 0,..., 0, \overset{j}{1}, 0,..., 0), \qquad j = 1,..., n.$$

To every word $\sigma_1 \cdots \sigma_l$ with l letters, we associate a walk with l steps:

$$(0, 0,..., 0) \to \mathbf{e}_{\sigma_1} \to \mathbf{e}_{\sigma_1} + \mathbf{e}_{\sigma_2} \to \mathbf{e}_{\sigma_1} + \mathbf{e}_{\sigma_2} + \mathbf{e}_{\sigma_3} \to \cdots \to \mathbf{e}_{\sigma_1} + \mathbf{e}_{\sigma_2} + \cdots + \mathbf{e}_{\sigma_l}.$$

* Current address: Department of Mathematics, University of Illinois, Urbana, IL 61801.

0022-247X/80/030192-08$02.00/0

For example, in N^3 the word 31223211 corresponds to the path

$$\begin{aligned}(0,0,0) &\to (0,0,1) \to (1,0,1) \to (1,1,1) \to (1,2,1)\\ &\to (1,2,2) \to (1,3,2) \to (2,3,2) \to (3,3,2).\end{aligned}$$

For a point $\mathbf{m} = (m_1, \ldots, m_n)$, it is well known and easy to see that there are

$$\binom{m_1 + \cdots + m_n}{m_1, \ldots, m_n} = \frac{(m_1 + \cdots + m_n)!}{m_1! \cdots m_n!}$$

possible ways of getting from the origin to that point.

In Section 1 we shall introduce "lanes" on certain roads which will enable us to interpret $q^{\mathrm{maj}(\sigma)}$, $q^{i(\sigma)}$ as the number of possible routes (counting lanes) to walk the path σ. This interpretation is then used to prove MacMahon's Theorems 3.6 and 3.7 in Andrews [1]. In Section 2 we rederive the generating function for the classical Eulerian polynomials $A_n(t)$ (Comtet [2], Foata and Schützenberger [3]) and in Section 3 Gessel's generating function for the "maj" q-Eulerian polynomials ${}^{\mathrm{maj}}A_n(t)$ (Garsia [4, I. 15]) is obtained using the present method. Finally it is indicated how to obtain Stanley's [5] generating function for ${}^{i}A_n(t)$ (Garsia [4, I.12]) and Gessel's generating function for the trivariate ${}^{m,i}A_n(t, p, q)$ (Garsia [4, (3.2)]).

The present method is not yet capable of considering the "statistics" $d(\sigma^{-1})$ and $m(\sigma^{-1})$ discussed in a recent paper of Garsia and Gessel. The reason is that we do not know how to interpret σ^{-1} in terms of a lattice walk. It would be interesting to extend the present method as to contain these "statistics" as well.

For the general framework underlying the present approach, we refer the reader to the excellent paper of Wilf [6].

1. Two Theorems of MacMahon

THEOREM 1 (Theorem 3.6 in Andrews [1, p. 41]). *Let $C(m_1, \ldots, m_n)$ denote the set of all walks from $(0, \ldots, 0)$ to $(m_1, \ldots, m_n)$ (alternatively the set of all words in $\{1, \ldots, n\}$ with m_1 1's, m_2 2's,..., m_n n's); then*

$$\sum q^{i(\sigma)} = \begin{bmatrix} m_1 + \cdots + m_n \\ m_1, \ldots, m_n \end{bmatrix}, \tag{1.1}$$

where the sum is taken over all walks σ in $C(m_1, \ldots, m_n)$ and the r.h.s. *is the q-multinomial coefficient $(q)_{m_1+\cdots+m_n}/(q)_{m_1} \cdots (q)_{m_n}$, where $(x)_a = (1-x)(1-qx) \cdots (1-q^{a-1}x)$.*

Proof. (We recommend that the reader first go through the proof with $n = 2$.) In the n-dimensional lattice N^n, for every point $\mathbf{m} = (m_1, \ldots, m_n)$ introduce $q^{m_1+\cdots+m_{i-1}}$ lanes in the block between $\mathbf{m} - \mathbf{e}_i$ and $\mathbf{m}$ (provided

$m_i > 0$), $i = 1, \ldots, n$. It is readily seen that for a walk σ, there are $q^{i(\sigma)}$ possible routes to travel it (counting lanes). Thus the l.h.s. of (1.1) is the total number of possible routes to get from $(0, 0, \ldots, 0)$ to $\mathbf{m} = (m_1, \ldots, m_n)$. Denoting this number by $F(\mathbf{m})$ we have that F satisfies the partial difference equation

$$F(\mathbf{m}) = \sum_{i=1}^{n} q^{m_1 + \cdots + m_{i-1}} F(\mathbf{m} - \mathbf{e}_i)$$

(since every path which terminates at $\mathbf{m}$ must come from $\mathbf{m} - \mathbf{e}_1, \mathbf{m} - \mathbf{e}_2, \ldots,$ or $\mathbf{m} - \mathbf{e}_n$).

Introducing the negative shift operators $E_i^{-1} F(\mathbf{m}) = F(\mathbf{m} - \mathbf{e}_i)$, $i = 1, \ldots, n$, the above equation can be written

$$\left(I - \sum_{i=1}^{n} q^{m_1 + \cdots + m_{i-1}} E_i^{-1}\right) F \equiv 0. \tag{1.2}$$

The function

$$G(\mathbf{m}) = \begin{bmatrix} m_1 + \cdots + m_n \\ m_1, \ldots, m_n \end{bmatrix}$$

is also satisfies by (1.2) (check !) and since $F \equiv G$ on $\bigcup_{i=1}^{n} \{m_i = 0\}$ by the natural inductive hypothesis, $F \equiv G$ throughout the whole lattice N^n.

THEOREM 2 (Theorem 3.7 in Andrews [1, p. 42]). *With the above notation*

$$\sum_{\sigma \in C(m_1, \ldots, m_n)} q^{\operatorname{maj}(\sigma)} = \begin{bmatrix} m_1 + \cdots + m_n \\ m_1, \ldots, m_n \end{bmatrix}. \tag{1.3}$$

Proof. For the sake of clarity we shall first carry the proof for $n = 2$.

In the lattice N^2 we introduce $q^{m_1 + m_2 - 1} - 1$ extra "express lanes" from $(m_1 - 1, m_2 - 1)$ to (m_1, m_2) via $(m_1 - 1, m_2)$, where one is not allowed to stop at $(m_1 - 1, m_2)$. Thus there are $q^{m_1 + m_2 - 1}$ ways of traveling the path $(m_1 - 1, m_2 - 1) \to (m_1 - 1, m_2) \to (m_1, m_2)$ while there is only one possible way of doing the journey $(m_1 - 1, m_2 - 1) \to (m_1, m_2 - 1) \to (m_1, m_2)$. It is readily seen that for a given walk, there are $q^{\operatorname{maj}(\sigma)}$ possible ways of traveling it. Thus the l.h.s. of (1.3), $F(m_1, m_2)$, describes the number of possible ways of traveling from $(0, 0)$ to (m_1, m_2).

$F(m_1, m_2)$ satisfies

$$\begin{aligned} F(m_1, m_2) = {} & F(m_1 - 1, m_2) + F(m_1, m_2 - 1) \\ & + (q^{m_1 + m_2 - 1} - 1) F(m_1 - 1, m_2 - 1). \end{aligned} \tag{1.4}$$

But

$$G(m_1, m_2) = \begin{bmatrix} m_1 + m_2 \\ m_1, m_2 \end{bmatrix}$$

is also satisfied by (1.4) and since $F(m_1, 0) = G(m_1, 0)$, $F(0, m_2) = G(0, m_2)$, it follows that $F(m_1, m_2) = G(m_1, m_2)$ for every m_1, m_2.

Proof for general n. Put $d = m_1 + \cdots + m_n$. For every $\mathbf{m} \in N^n$ and every subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$, where $i_1 > i_2 > \cdots > i_k$, and $m_{i_1} > 0, \ldots, m_{i_k} > 0$, introduce $(q^{d-1} - 1)(q^{d-2} - 1) \cdots (q^{d-k+1} - 1)$ "express lanes" with the path

$$(\mathbf{m} - \mathbf{e}_{i_1} - \cdots - \mathbf{e}_{i_k}) \to (\mathbf{m} - \mathbf{e}_{i_2} - \cdots - \mathbf{e}_{i_k}) \to \cdots \to \mathbf{m},$$

where one is not allowed to stop in the intermediate points. There are altogether $q^{d-1}q^{d-2} \cdots q^{d-k+1}$ possible ways of doing the above journey where one is allowed to stop (check!). For a given walk σ it is seen that there are $q^{\mathrm{maj}(\sigma)}$ possible routes of doing the path (counting lanes) and the l.h.s. of (1.3) counts all the possible ways of getting from $\mathbf{0}$ to $\mathbf{m}$. Denoting this number by $H(\mathbf{m})$, we get that H satisfies the partial difference equation

$$\begin{aligned} H(\mathbf{m}) = {} & \sum_{i=1}^{n} H(\mathbf{m} - \mathbf{e}_i) + (q^{d-1} - 1) \sum_{i \neq j} H(\mathbf{m} - \mathbf{e}_i - \mathbf{e}_j) \\ & + (q^{d-1} - 1)(q^{d-2} - 1) \sum_{i \neq j \neq k} H(\mathbf{m} - \mathbf{e}_i - \mathbf{e}_j - \mathbf{e}_k) \\ & + \cdots + (q^{d-1} - 1) \cdots (q^{d-n+1} - 1) H(m_1 - 1, m_2 - 1, \ldots, m_n - 1), \end{aligned} \tag{1.5}$$

which in operator notation is

$$q^{m_1 + \cdots + m_m} H = \prod_{i=1}^{n} (I + (q^{m_1 + \cdots + m_m} - 1) E_i^{-1}) H, \tag{1.6}$$

or

$$\begin{aligned} q^{m_1 + \cdots + m_m} H = {} & \Big[I + (q^d - 1) \sum_{i=1}^{n} E_i^{-1} + (q^d - 1)(q^{d-1} - 1) \sum_{i \neq j} E_i^{-1} E_j^{-1} \\ & + \cdots + (q^d - 1)(q^{d-1} - 1) \cdots (q^{d-k+1} - 1) \sum_{i_1 \neq \cdots \neq i_k} E_{i_1}^{-1} \cdots E_{i_k}^{-1} \\ & + \cdots + (q^d - 1) \cdots (q - 1) E_1^{-1} E_2^{-1} \cdots E_n^{-1} \Big] H. \end{aligned} \tag{1.7}$$

But

$$G(\mathbf{m}) = \begin{bmatrix} m_1 + \cdots + m_n \\ m_1, \ldots, m_n \end{bmatrix}$$

is also a solution of this partial difference equation (check!) and $H = G$ on $\bigcup_{i=1}^{n} \{m_i = 0\}$ by the natural inductive hypothesis (i.e., by the theorem for $n - 1$); thus $H(\mathbf{m}) = G(\mathbf{m})$ for every $\mathbf{m} \in N^n$.

2. Eulerian Polynomials

Let

$$F_d(m_1, \ldots, m_n) = \sum_{\sigma \in C(m_1, \cdots, m_n)} t^{1+d(\sigma)}.$$

Then, since $S_n = C(1, 1, \ldots, 1)$ we have $A_n(t) = F_d(1, 1, \ldots, 1) = B(n)$, say. We set $A_0(t) = B(0) = t$. The same reasoning as that in the proof of Theorem 2, only simpler, yields that $F_d(\mathbf{m})$ is a solution of the partial difference equation

$$tF_d(\mathbf{m}) = \prod_{i=1}^{n} (I + (t-1) E_i^{-1}) F_d(\mathbf{m}). \tag{2.1}$$

Substituting $\mathbf{m} = (1, \ldots, 1)$ and noting that $F_d(\mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_k}) = B(k)$, when $i_1 \neq \cdots \neq i_k$, we get

$$tB(n) = (I - (1-t) E^{-1})^n B(n)$$

(where $E^{-k}B(n) = B(n-k)$). Thus

$$tB(n) = \sum_{k=1}^{n} (-1)^{n-k} (1-t)^{n-k} \binom{n}{k} B(k), \qquad n \geqslant 1,$$

or

$$\frac{tB(n)}{n!(1-t)^{n+1}} = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!} \frac{B(k)}{k!(1-t)^{k+1}}, \qquad n \geqslant 1. \tag{2.2}$$

Multiplying (2.2) by u^n and summing up yield

$$(t - e^{-u}) \sum_{0} \frac{B(n)\, u^n}{n!(1-t)^{n+1}} = -t$$

and so

$$\sum \frac{A_n(t)\, u^n}{n!(1-t)^{n+1}} = \frac{te^u}{1-te^u} \qquad \text{(I.6 in Garsia [4]).}$$

Exercise. Using (2.1) obtain the generating function of $F_d(m_1, \ldots, m_n)$ and deduce the solution to Simon Newcomb's problem.

3. q-Eulerian Polynomials

3.1. The above method generalizes to give generating functions to the q-Eulerian polynomials

$${}^{i}A_n(t) = \sum_{\sigma \in S_n} q^{i(\sigma)} t^{1+d(\sigma)},$$

$${}^{\mathrm{maj}}A_n(t) = \sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)} t^{1+d(\sigma)},$$

and

$$^{i,m}A_n(t, p, q) = \sum_{\sigma \in S_n} p^{i(\sigma)} q^{\mathrm{maj}(\sigma)} t^{d(\sigma)}$$

(see Garsia [4]). We shall prove (I.15 in Garsia [4]) Gessel's formula

$$\sum_{n \geqslant 0} \frac{u^n}{n!} \frac{^{\mathrm{maj}}A_n(t)}{(1-t)(1-qt)\cdots(1-q^n t)} = \sum_{k \geqslant 1} t^k e^{u(1+q+\cdots+q^{k-1})} \tag{3.1}$$

and sketch the derivation of the other generating functions.

Consider

$$F_{d,\mathrm{maj}}(m_1, \ldots, m_n) = \sum_{\sigma \in C(m_1, \ldots, m_n)} q^{\mathrm{maj}(\sigma)} t^{1+d(\sigma)};$$

then the same reasoning which lead to formula (1.7) leads to (recall $d = m_1 + \cdots + m_n$)

$$\begin{aligned} tq^d F_{d,\mathrm{maj}} \equiv \Big[& I + (tq^d - 1) \sum_{i=1}^{n} E_i^{-1} + (tq^d - 1)(tq^{d-1} - 1) \sum_{i \neq j} E_i^{-1} E_i^{-1} \\ & + \cdots + (tq^d - 1)(tq^{d-1} - 1) \cdots (tq^{d-k+1} - 1) \sum_{i_1 \neq \cdots \neq i_k} E_{i_1}^{-1} \cdots E_{i_k}^{-j} \\ & + \cdots + (tq^d - 1) \cdots (tq - 1) E_1^{-1} E_2^{-1} \cdots E_n^{-1} \Big] F_{d,\mathrm{maj}} . \end{aligned} \tag{3.2}$$

Now, $^{\mathrm{maj}}A_n(t) = F_{d,\mathrm{maj}}(1, 1, 1, \ldots, 1)$, and using (3.2) at the point $(1, 1, 1, \ldots, 1)$, i.e., $m_1 = m_2 = \cdots = m_n = 1$, and setting $B(k) = {}^{\mathrm{maj}}A_k(t)$, one gets

$$tq^n B(n) = \sum \binom{n}{k} (tq^n - 1) \cdots (tq^{n-k+1} - 1) B(k),$$

so

$$\frac{tq^n B(n)}{n!(qt)_n} - \sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!} \frac{B(k)}{k!(qt)_k} = 0, \qquad n \geqslant 1. \tag{3.3}$$

Multiplying by u^n, $n = 0, 1, 2, \ldots$, and summing yield

$$t \sum_0^{\infty} \frac{q^n u^n B(n)}{n!(qt)_n} - e^{-u} \sum_0^{\infty} \frac{B(n)\, u^n}{n!(qt)_n} = (t-1)\, t.$$

Let

$$z(u) = \frac{1}{1-t} \sum_0^{\infty} \frac{B(n)}{n!(qt)_n} u^n \qquad (= \text{l.h.s. of (3.1)});$$

then

$$tz(qu) - e^{-u}z(u) = -t.$$

Introducing the q dilation operator $Qf(u) = f(qu)$ we have $(tQ - e^{-u})\, z(u) = -t$. Thus (3.1) follows:

$$\begin{aligned} z(u) &= t(e^{-u} - tQ)^{-1}\, 1 = t[e^{-u}(1 - te^{u}Q)]^{-1}\, 1 \\ &= t(1 - te^{u}Q)^{-1}\, e^{u} = t \sum_{0}^{\infty} t^{k}(e^{u}Q)^{k}\, e^{u} \\ &= \sum_{k \geqslant 1} t^{k} e^{u[1+q+\cdots+q^{k-1}]}. \end{aligned}$$

3.2. Defining $F_{d,i}(m_1, \ldots, m_n)$ similarly one sees that it satisfies the partial difference equation

$$tF_{d,i} \equiv \prod_{i=1}^{n} [1 + (t-1)\, q^{m_1+\cdots+m_{i-1}} E_i^{-1}]\, F_{d,i}\,.$$

After some manipulations one gets that $A(n) = {}^{i}A_n(t) = F_{d,i}(1, 1, 1, \ldots, 1)$ satisfies

$$\frac{tA(n)}{(q)_n} = \sum \frac{(t-1)^{n-k}}{(q)_{n-k}} \frac{A(k)}{(q)_k}, \qquad n = 1, 2, \ldots, \tag{3.2.1}$$

from which Stanley's [5] (Garsia [4, I.12]) generating function follows. Finally

$$F_{d,i,\mathrm{maj}}(m_1, \ldots, m_n) = \sum_{\sigma \in C(m_1, \ldots, m_n)} p^{i(\sigma)} q^{\mathrm{maj}(\sigma)} t^{d(\sigma)+1}$$

satisfies

$$tq^{m_1+\cdots+m_m} F = \prod_{i=1}^{n} [1 + (tq^{m_1+\cdots+m_m} - 1)\, p^{m_1+\cdots+m_{i-1}} E_i^{-1}],$$

which implies, after some manipulations, that $A_n(t, p, q) = F_{d,i,\mathrm{maj}}(1, 1, \ldots, 1)$ satisfies

$$\frac{tq^n A(n)}{[n]_p\, (qt)_n} = \sum \frac{(-1)^{n-k}}{[n-k]_p} \frac{A(k)}{[k]_p\, (qt)_k}$$

(here $[l]_p = (1 + p) \cdots (1 + p + \cdots + p^{l-1})$), from which Gessel's generating function for $A_n(t, p, q)$ follows (Garsia [4, formula 3.2]).

ACKNOWLEDGMENTS

The author wishes to thank Saj-Nicole A. Joni for introducing him to the fascinating subject of q-combinatorics, and for stimulating conversations. He is also grateful to Richard J. Duffin for introducing him to S. A. Joni.

REFERENCES

1. G. E. ANDREWS, The theory of partitions, *in* "Encyclopedia of Math. and its Applications," Vol. 2, Addison-Wesley, Reading, Mass., 1976.
2. L. COMTET, "Advanced Combinatorics," Reidel, Dordrecht, 1970.
3. D. FOATA AND M. P. SCHÜTZENBERGER, "Théorie géométrique des Polynomes Eulériens," Springer Lecture Notes No. 138, Springer-Verlag, Berlin/New York, 1970.
4. A. M. GARSIA, On the "maj" and "inv" q-analogues of Eulerian polynomials, *J. Linear and Multilinear Algebra* (to appear).
5. R. P. STANLEY, Binomial posets, Möbius inversion and permutation enumeration, *J. Combinatorial Theory Ser. A* 20, No. 3 (1976), 336–356.
6. HERBERT S. WILF, A unified setting for sequencing, ranking, and selection algorithms for combinatorial objects, *Advances in Math.* 24 (1977), 281–291.