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PARTIAL DIFFERENCE EQUATIONS IN $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$ AND THEIR APPLICATIONS TO COMBINATORICS

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Various discrete functions encountered in Combinatorics are solutions of Partial Difference Equations in the subset of $\mathbf{N}^n$ given by $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$. Given a partial difference equation, it is described how to pass from the standard "easy" solution of an equation in $\mathbf{N}^n$ to a solution of the same equation subject to certain "Dirichlet" or "Neumann" boundary conditions in the domain $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$ and related domains. Applications include a rather quick derivation of MacMahon's generating function for plane partitions, a generalization and q-analog of the Ballot problem, and a joint analog of the Ballot problem and Simon Newcomb's problem.

0. Introduction

There is a very close analogy between Physics and Enumerative Combinatorics. The former is often looking for solutions of partial differential equations in a given region of $\mathbf{R}^n$ under prescribed boundary conditions, while the latter is seeking solutions of partial difference equations in certain subsets of $\mathbf{N}^n$. While nobody ever dared object the O.K. ness of using PDEs in Physics, the use of partial difference equations in Combinatorics was at best tolerated as a temporary nuisance to be put up with until a "direct combinatorial proof" was found. The use of partial difference equations received the derogatory names: "inductive", "recurrence", and G.H. Hardy even called it "essentially verifications" (quoted by Andrews [1, p. 105]).

Although some of our favorite proofs are "direct combinatorial", they are not any better, as a whole, than recurrence proofs. Indeed, very few proofs beat the elegance of Good's [5] proof of Dyson's conjecture and Moon's [9, p. 13] proof for the number of labelled trees, both of which use recurrence.

Among the few who did not have any scruples using partial difference equations was the great MacMahon. His solution of the Ballot problem and his derivation of generating functions for plane partitions, together with other problems, all employed partial difference equations. It may be that his lengthy and ad-hoc ways of solving these partial difference equations was one of the reasons which gave difference equations their bad name. Surprisingly enough, a small change in the formulation of the boundary conditions could have saved him a lot of trouble.

In the present paper we introduce this change and use the algebra of partial difference operators (known but unexploited by MacMahon), to rederive the solution of the Ballot problem and generating functions for plane partitions, together with various generalizations.

Stanley [10, pp. 259, 269] is baffled by the fact that although MacMahon's generating function for plane partitions $\prod_{i=1}^{\infty}(1-q^k)^{-k}$ is so simple, its proof is indirect and rather complicated. We believe that our proof gives a more or less "obvious" reason why this formula is so simple.

Consider the two dimensional Ballot problems, i.e., finding the number of ways, $F(m_1, m_2)$, of walking with positive unit steps, from $(0, 0)$ to (m_1, m_2), without ever crossing the diagonal $\{m_1 = m_2\}$. MacMahon [7, p. 127] has set the partial difference equation

$$F(m_1 m_2) = F(m_1 - 1, m_2) + F(m_1, m_2 - 1), (m_1 > m_2) \tag{0.1}$$

together with the boundary conditions

$$F(m_1, m_2) = F(m_1, m_2 - 1) \quad \text{on} \quad m_1 = m_2. \tag{0.2}$$

MacMahon's stumbling block was the fact that $F(m_1, m_2)$ is not defined for $m_1 < m_2$. However, by extending F to $\{m_1 - m_2 > -1\}$ and requiring $F(m_1, m_2) = 0$ on its boundary $\{m_1 - m_2 = -1\}$ as indeed it should be, (0.1) can be required to hold in $m_1 \geqslant m_2$:

$$F(m_1, m_2) = F(m_1 - 1, m_2) + F(m_1, m_2 - 1), m_1 - m_2 \geqslant 0 \tag{0.1$'$}$$

and then (0.2) implies

$$F(m_1, m_2) = 0 \quad \text{on} \quad m_1 - m_2 = -1. \tag{0.2$'$}$$

Borrowing terminology from PDE, we replaced the "Neumann" problem (0.1), (0.2) by an easier "Dirichlet" problem (0.1′), (0.2′). In the case $n = 2$, very little is gained by this modification, but the analogous modification for $n > 2$ makes life so much easier.

It is not clear how MacMahon solved the Ballot problem for $n > 3$. In [7, p. 127–133] he solves it for $n = 2, 3$ and then goes on to state the general case. It is possible that he simply extrapolated from $n = 2, 3$ to the general case without bothering to prove the resulting formula. Be that as it may, our Theorem 5 closes this gap. (It should be noted that there exist several other proofs of this result.)

Next let us describe the content. Section 1 introduces the nomenclature of Partial Difference Operators and considers lattice walks. This is illustrated by the ordinary lattice walk, Simon Newcomb's problem, and the generating function for the lesser index of a walk. Section 2 gives a general solution of a Dirichlet problem in the region $\bigcap_{i=1}^{n-1}\{m_i - m_{i+1} \geqslant -1\} \cap (m_n \geqslant 0)$ and related regions, for a wide class of partial difference equations.

Section 3 present applications to the Ballot problem, and its generalization and q-analog, and to the restricted Simon Newcomb problem. Sections 4 and 5 treat classical plane partitions while Section 6 gives further applications to what we coined "pseudo-plane partitions". The Andrews–Gordon–MacDonald theorem (nèe Bender–Knuth conjecture) (Stanley [10, p. 265]) resisted all our attempts at an easy solution. It is hoped that in the future the present method will be capable of providing such a proof.

Before closing this introduction we should mention Carlitz's [3] (see also Andrews [1, p. 180–184]) elegant recurrence proof of MacMahon's determinant formula and the subsequent derivation of MacMahon's generating function. Our novelty is in dispensing with determinants altogether. Recently a very elegant and general determinant formula encompassing both lattice walks and general plane partitions has been done by Gessel [4] (which we can't help admiring even though he belongs to the "direct combinatorial" enemy camp).

1. Partial difference operators and lattice walk

1.1. Let $\mathbf{Z}=\{0, \pm 1, \pm 2, \ldots\}$, $\mathbf{N}=\{0, 1, 2, 3, \ldots\}$. Consider functions $f: \mathbf{Z}^n \to \mathbf{C}$, $f(m)=f(m_1, \ldots, m_n)$, and define the fundamental shift operators

$$X_i^{-1} f(m_1, \ldots, m_i, \ldots, m_n) = f(m_1, \ldots, m_i - 1, \ldots, m_n),$$

$i=1, \ldots, n$. Denoting by $\boldsymbol{e}_i$ the unit vector in the m_i coordinate, the above is shorthanded to $X_i^{-1} f(\boldsymbol{m}) = f(\boldsymbol{m}-\boldsymbol{e}_i)$. For $\boldsymbol{\alpha} \in \mathbf{N}^n$ we write $X^{-\boldsymbol{\alpha}} = X_1^{-\alpha_1} \cdots X_n^{-\alpha_n}$ and so $X^{-\boldsymbol{\alpha}} f(\boldsymbol{m}) = f(\boldsymbol{m}-\boldsymbol{\alpha})$. A typical linear partial difference operator has the form

$$P = \sum_{\substack{\boldsymbol{\alpha} \geq 0 \\ |\boldsymbol{\alpha}| \leq M}} a_{\boldsymbol{\alpha}} X^{-\boldsymbol{\alpha}},$$

where $a_{\boldsymbol{\alpha}} = a_{\boldsymbol{\alpha}}(\boldsymbol{m})$ are discrete functions, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$ and $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n$. If $a_0(\boldsymbol{m}) \neq 0$ for every $\boldsymbol{m}$, P is said to be hyperbolic and then we can assume that $a_0 = 1$. We shall be concerned with solving certain partial difference equations, $Pf = 0$, in certain subsets of $\mathbf{N}^n$.

Define the *discrete delta function* δ by

$$\delta(\boldsymbol{0}) = 1, \qquad \delta(\boldsymbol{m}) = 0, \quad \boldsymbol{m} \neq \boldsymbol{0}.$$

Definition. A function $f: \mathbf{Z}^n \to \mathbf{C}$ satisfying $Pf = \delta$ is called a *fundamental solution* corresponding to the operator P. If f is supported in $\mathbf{N}^n$, it is called a *canonical fundamental solution.*

Proposition 1. *A particle starts at the origin and has $a_{\boldsymbol{\alpha}}(m)$ ways of jumping from $\boldsymbol{m}-\boldsymbol{\alpha}$ to $\boldsymbol{m}$, where $\boldsymbol{\alpha} \in \mathbf{N}^n$. Let $F(\boldsymbol{m})$ be the total number of ways of getting from the origin to $\boldsymbol{m}$. Then $F(\boldsymbol{m})$ is the canonical fundamental solution corresponding to $I - \sum a_{\boldsymbol{\alpha}} X^{-\boldsymbol{\alpha}}$.*

Proof. Since there is no way of getting out of $\mathbf{N}^n$, F vanishes outside $\mathbf{N}^n$. Consider all the walks terminating at $\boldsymbol{m}$. The particle's last stop was $\boldsymbol{m}-\boldsymbol{\alpha}$, for some $\boldsymbol{\alpha}$, and this contributes $a_{\boldsymbol{\alpha}}(\boldsymbol{m})\ F(\boldsymbol{m}-\boldsymbol{\alpha})$ ways. If $\boldsymbol{m}-\boldsymbol{\alpha}\notin\mathbf{N}^n$ then there is no contribution and $F(\boldsymbol{m}-\boldsymbol{\alpha})=0$.

Thus

$$F(\boldsymbol{m})-\sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}}F(\boldsymbol{m}-\boldsymbol{\alpha})=0, \qquad \boldsymbol{m}\in\mathbf{N}^n, \quad \boldsymbol{m}\neq 0. \tag{1.1}$$

If $\boldsymbol{m}=\boldsymbol{0}$, the r.h.s. of (1.1) is 1. If $\boldsymbol{m}\notin\mathbf{N}^n$, everything is zero. It follows that

$$\left(I-\sum a_{\boldsymbol{\alpha}}X^{-\boldsymbol{\alpha}}\right)F=\delta.$$

Since F is supported in $\mathbf{N}^n$ the proof is completed. □

From now on we shall focus attention on equation $Pf=0$, where P has the special form ($d=m_1+\cdots+m_n$):

$$P=I-a_1(d)\sum_{i=1}^{n} X_i^{-1}-a_2(d)\sum_{i=j} X_i^{-1}X_j^{-1}-\cdots-a_n(d)X_1^{-1}\cdots X_n^{-1}.$$

Note that since P is symmetric, its canonical fundamental solution is a symmetric function.

1.2. Generating functions. To every function $f:\mathbf{N}^n\to\mathbf{C}$ corresponds the generating function $\hat{f}(\boldsymbol{z})=\sum f(\boldsymbol{m})\boldsymbol{z}^{\boldsymbol{m}}$. Let P be an operator with constant coefficients $P(X_1^{-1},\ldots,X_n^{-1})$. Since $[X^{-\boldsymbol{\alpha}}f]\hat{}=\boldsymbol{z}^{\boldsymbol{\alpha}}\hat{f}$, we have $[P(X_1^{-1},\ldots,X_n^{-1})f]\hat{}=P(z_1,\ldots,z_n)\hat{f}$. If, in addition, P is hyperbolic, i.e., the constant term is non-zero, then $Pf=\delta$ implies $P\hat{f}=1$ and $\hat{f}=1/P(z_1,\ldots,z_n)$.

1.3. Examples. (i) Ordinary positive lattice walk: $P=1-X_1^{-1}-\cdots-X_n^{-1}$, whose canonical fundamental solution is

$$F(\boldsymbol{m})=\frac{(m_1+\cdots+m_n)!}{m_n!\cdots m_n!} \quad\text{and}\quad \hat{F}=\frac{1}{(1-z_1-\cdots-z_n)}.$$

(ii) Let $C(m_1,\ldots,m_n)$ be the set of words in the alphabet $\{1,\ldots,n\}$ with m_1 1's, m_2 2's, $\ldots$, m_n n's (there is a one-one correspondence between $C(m_1,\ldots m_n)$ and the set of paths from $\boldsymbol{0}$ to $(m_1,\ldots,m_n)$). For a word $\sigma=\sigma_1\cdots\sigma_k$ define

$$a(\sigma)=\sum_{i=1}^{k-1}\chi(\sigma_i<\sigma_{i+1})$$

(here $\chi(A)=1$, if A is true; $\chi(A)=0$, if A is false); $a(\sigma)$ is called the number of *ascents* of the word σ. Simon Newcomb's problem consider $F_t(\boldsymbol{m})=\sum_{\sigma\in C(\boldsymbol{m})}t^{a(\sigma)}$. It is well known (e.g., Zeilberger [11]) that $tF_t=(\prod_{i=1}^{n}[I+(t-1)X_i^{-1}])F_t$. This means that F_t is the canonical fundamental solution of $(t-1)^{-1}[t-$

$\prod_{i=1}^{n}(I+(t-1)X_i^{-1})]$. Thus the generating function of F_t is given by

$$\hat{F}_t=(t-1)\Big/\left[t-\prod_{i=1}^{n}(1+(t-1)z_i)\right].$$

Remark. For reasons to become clear later, we prefer to work with *ascents* rather than the more customary descents. The theories are equivalent.

(iii) The *lesser index* of a word $\sigma=\sigma_1\cdots\sigma_k$ is defined (MacMahon [7, p. 136]) by

$$r(\sigma)=\sum_{i=1}^{k-1} i\chi(\sigma_i<\sigma_{i+1})$$

i.e., the sum of places where ascents occur. Consider $F(\boldsymbol{m})=\sum_{\sigma\in C(\boldsymbol{m})} q^{r(\sigma)}$. In Zeilberger [11] it was shown that $F(\boldsymbol{m})$ is the canonical fundamental solution corresponding to

$$P=I-\sum_{i=1}^{n} X_i^{-1}$$
$$-(q^d-1)\sum_{i\neq j} X_i^{-1}X_j^{-1}-\cdots-(q^d-1)\cdots(q^{d-n+1}-1)X_1^{-1}\cdots X_n^{-1}.$$

F is given by the q-multinomial coefficients.

$$F(m_1,\ldots,m_n)=(q)_{m_1+\cdots+m_n}/(q)_{m_1}\cdots(q)_{m_n},$$

where $(x)_a=(1-x)(1-qx)\cdots(1-q^{a-1}x)$.

Remark. We prefer to work with the lesser index rather than the major index. The theories of these two indices are equivalent.

2. Solutions of a boundary value problem in $\bigcap_{i=1}^{n-1}\{m_i-m_{i+1}\geqslant -1\}\cap\{m_n\geqslant 0\}$

Theorem 2. *Let $d=m_1+\cdots+m_n$, and let $F_n(\boldsymbol{m})=F_n(m_1,\ldots,m_n)$ be the canonical fundamental solution corresponding to*

$$P=I-a_1(d)\sum_{i=1}^{n} X_i^{-1}-a_2(d)\sum_{i\neq j} X_i^{-1}X_j^{-1}-\cdots-a_n(d)X_1^{-1}\cdots X_n^{-1}.$$

This means that $PF_n=\delta$, and F_n vanishes outside $\mathbf{N}^n$. Let G_n be the unique solution of the equation $Pf=0$ in $\{m_1-m_2\geqslant -1\}\cap\{m_2-m_3\geqslant -1\}\cap\cdots\cap\{m_{m-1}-m_n\geqslant -1\}$, subject to the boundary conditions

$$f=0\quad\text{on}\ \{m_1=m_2-1\}\cup\{m_2=m_3-1\}\cup\cdots\cup\{m_{n-1}=m_n-1\}$$

and

$$f(m_1,\ldots,m_{n-1},0)=G_{n-1}(m_1,\ldots,m_{n-1}),\qquad G_1(m_1)\equiv 1.$$

Then G_n is given by

$$G_n = \prod_{1 \le i < j \le n} (1 - X_i X_j^{-1}) F_n. \tag{2.1}$$

We are going to need the following simple lemma,

Lemma 3. *Let $H: \mathbf{Z}^2 \to \mathbf{C}$ satisfy $H(l_1, l_2) = H(l_2, l_1)$, then $(1 - X_1 X_2^{-1}) H(l_1, l_2)$ vanishes on $l_1 = l_2 - 1$. More generally $(1 - X_1^k X_2^{-k}) H(l_1, l_2)$ vanishes on $l_1 = l_2 - k$.*

Proof of Theorem. 2. Since $X_i X_j^{-1}$ commutes with the operator P, so does $\prod_{i<j} (1 - X_i X_j^{-1})$. Since $F_n(\boldsymbol{m})$ is a solution of $Pf = 0$ in the interior of $\mathbf{N}^n$, so is $G_n(\boldsymbol{m})$. It remains to verify that $G_n(\boldsymbol{m})$ satisfies the prescribed boundary conditions. Now,

$$\begin{aligned} G_n(\boldsymbol{m}) = (1 - X_1 X_2^{-1}) [&(1 - X_1 X_3^{-1})(1 - X_1 X_4^{-1}) \cdots (1 - X_1 X_n^{-1}) \\ &(1 - X_2 X_3^{-1})(1 - X_2 X_4^{-1}) \cdots (1 - X_2 X_n^{-1}) \\ &\vdots \\ &(1 - X_{n-1} X_n^{-1})] F_n(\boldsymbol{m}). \end{aligned}$$

The operator inside the square brackets is symmetric with respect to X_1, X_2, and since $F_n(\boldsymbol{m})$ is a symmetric function, we can write

$$G_n(m_1, m_2) = (1 - X_1 X_2^{-1}) H(m_1, m_2),$$

where $H(m_1, m_2) = H(m_2, m_1)$ and the dependence on $m_3, \ldots, m_n$ is suppressed. It follows by Lemma 3 that $G_n(\boldsymbol{m}) = 0$ on $m_1 = m_2 - 1$. Similarly, for $i = 1, \ldots, n-1$

$$G_n(\boldsymbol{m}) = (1 - X_i X_{i+1}^{-1})[\text{operator symmetric w.r.t. } X_i, X_{i+1}] F_n(\boldsymbol{m}),$$

and so $G_n(\boldsymbol{m}) = 0$ on $m_i = m_{i+1} - 1$. Finally on $m_n = 0$,

$$\begin{aligned} G_n(m_1, \ldots, m_{n-1}, 0) = &\prod_{i \le i < j \le n-1} (1 - X_i X_j^{-1}) \\ &\times [1 - X_1 X_n^{-1}) \cdots (1 - X_{n-1} X_n^{-1})] F_n(m_1, \ldots, m_{n-1}, 0). \end{aligned}$$

But $F_n(\boldsymbol{m}) = 0$ for $m_n < 0$, so

$$\begin{aligned} G_n(m_1, \ldots, m_{n-1}, 0) &= \prod_{i \le i < j \le n-1} (1 - X_i X_j^{-1}) F_{n-1}(m_1, \ldots, m_{n-1}) \\ &= G_{n-1}(m_1, \ldots, m_{n-1}). \quad \square \end{aligned}$$

Similarly, using the second part of Lemma 3, we can prove,

Theorem 4. *Let P and F_n be as in Theorem 2. Let k be a positive integer and let H_n be the unique solution of the partial difference equation $Pf = 0$ in*

$$\{m_1 - m_2 \ge -k\} \cap \{m_2 - m_3 \ge -k\} \cap \cdots \cap \{m_{n-1} - m_n \ge -k\} \cap \{m_n \ge 0\},$$

where k is some positive integer, subject to the boundary conditions

$$f=0 \quad \text{on } \{m_1=m_2-k\}\cup\{m_2=m_3-k\}\cup\cdots\cup\{m_{n-1}=m_n-k\}$$

and

$$f(m_1,\ldots,m_{n-1},0)=H_{n-1}(m_1,\ldots,m_{n-1}), \qquad H_1(m_1)\equiv 1.$$

Then H_n is given by

$$H_n=\prod_{1\le i<j\le n}(1-X_i^kX_j^{-k})F_n. \tag{2.2}$$

3. Applications to Ballot problems

3.1. Definition (MacMahon [7]). Let $f:A\to\mathbf{C}$, where $A\subset\mathbf{N}^n$. $\sum a(\boldsymbol{m})z^m$ is a redundant generating function for f if $f(\boldsymbol{m})=a(\boldsymbol{m})$ for $\boldsymbol{m}\in A$.

Theorem 5. (MacMahon [7, p. 133]). *Let $f_{\boldsymbol{m}}$ be the number of positive lattice paths from $\boldsymbol{0}$ to $\boldsymbol{m}$ where travel is restricted to the region $m_1\ge m_2\ge\cdots\ge m_n\ge 0$. $\prod_{i\le i<j\le n}(1-z_jz_i^{-1})/(1-z_1-\cdots-z_n)$ is a redundant generating function for $f_{\boldsymbol{m}}$ and*

$$f_{\boldsymbol{m}}=(m_1+\cdots+m_n)!\prod_{i\le i<j\le n}(m_i-m_j+j-i)/(m_1+n-1)!\cdots m_n!. \tag{3.1}$$

Proof. We apply Theorem 2 with $P=1-X_1^{-1}-\cdots-X_n^{-1}$,

$$F_n(m_1,\ldots,m_n)=(m_1+\cdots+m_n)!/m_1!\cdots m_n!$$

(see Example 1.3(i)); thus

$$f_{\boldsymbol{m}}=\prod_{1\le i<j\le n}(1-X_iX_j^{-1})[(m_1+\cdots+m_n)!/m_1!\cdots m_n!], \tag{3.2}$$

from which we get the redundant generating function.

Now $(m_1+\cdots+m_n)!$ commutes with the operator on the r.h.s. of (3.2) and so

$$G_n(m_1,\ldots,m_n)=(m_1+\cdots+m_n)!\prod_{i<j}(1-X_iX_j^{-1})[1/m_1!\cdots m_n!]$$

Consider

$$\begin{aligned}H(m_1,\ldots,m_n)&=\prod(1-X_iX_j^{-1})[1/m!\cdots m_n!]\\&=\prod_{1\le i<j\le n}(X_i^{-1}-X_j^{-1})\\&\quad\times[1/(m_1+n-1)!\,(m_2+n-2)!\cdots(m_{n-1}+1)!\,m_n!].\end{aligned}$$

It is seen that H is an alternating function (i.e., symmetric up to sign) of $m_1+n-1,\ m_2+n-2,\ldots,m_i+n-i,\ldots,m_n$. Also

$$H(m_1,\ldots,m_n)=Q(m_1+n-1,m_2+n-2,\ldots,m_n)/[(m_1+n-1)!\cdots m_n!],$$

where Q is an alternating polynomial of degree $n-1$ in each of its variables. But G, and therefore H, and therefore Q, vanish on $m_1=m_2-1$, $m_2=m_3-1,\ldots,m_{n-1}=m_n-1$; hence (m_1-m_2+1), $(m_2-m_3+1),\ldots,(m_{n-1}-m_n+1)$, are factors. By symmetry $([m_i+n-i]-[m_j+n-j]) = (m_i-m_j+j-i)$ are all factors, $1\leqslant i<j\leqslant n$; the theorem follows. □

Remark. The Ballot problem has an equivalent formulation in terms of Standard Young Tableaux and formula (3.1) easily implies the Frame–Robinson–Thrall formula, involving the hook lengths of the partition $m_1+m_2+\cdots+m_n$. We refer the reader to Greene–Nijenhuis–Wilf [6] where (s)he will find a very cute probabilistic proof of the F–R–T formula.

3.2. The political significance of the next theorem is in enumerating the total number of ways of counting votes such that at no time did a candidate lag by more than $(k-1)$ votes from the person destined to be immediately below (see Barton and Mallows [2, p. 243], where a solution to a more general problem is given in terms of a determinant. Our method also yields their result).

Theorem 6. *Let $F_k(\boldsymbol{m})$ be the number of lattice paths (with unit positive steps) from $\boldsymbol{0}$ to $\boldsymbol{m}$ such that one stays in the region $\prod_{i=1}^{n-1}\{m_i-m_{i+1}>-k\}$. A redundant generating function for F_k is*

$$\prod_{1\leqslant i<j\leqslant n}(1-z_j^k z_i^{-k})/(1-z_1-\cdots-z_n).$$

Proof. Apply Theorem 4 and use the remarks made on generating functions in Section 1.2.

3.3. Simon Newcomb's problem (MacMahon [7, p. 187]) considers the problem of counting the number of words in $1^{m_1}\cdots n^{m_n}$ with so and so many ascents (i.e., occurrences of $i\ j$ with $i<j$). In terms of walks in $\mathbf{N}^n$, calling all lines parallel to the m_i axis $(i=1,\ldots,n)$ "roads of kind i", Simon Newcomb asks for the number of walks from $\boldsymbol{0}$ to $\boldsymbol{m}$ with a specified number of "turns for the better", (in N.Y.C., $n=1$, roads of kind $1=$ "streets", roads of kind $2=$ "avenues"). Let us ask the same question for walks in which travel is restricted to

$$\bigcap_{i=1}^{n-1}\{m_i-m_{i+1}\geqslant 0\}.$$

Theorem 7. *Let $B(\boldsymbol{m};k)$ be the number of positive lattice walks from $\boldsymbol{0}$ to $\boldsymbol{m}\,(m_1\geqslant m_2\geqslant\cdots\geqslant 0)$ inside $\bigcap_{i=1}^{n-1}\{m_i-m_{i+1}\geqslant 0\}$, with k ascents. $B(\boldsymbol{m};k)$ is the coefficient of $z_1^{m_1}\cdots z_n^{m_n}t^k$ in*

$$\prod_{1\leqslant i<j\leqslant n}(1-z_jz_i^{-1})(t-1)\Big/\left[t-\prod_{i=1}^{n}(1+(t-1)z_i)\right].$$

Proof. Let $F(\boldsymbol{m})=\sum B(\boldsymbol{m};k)t^k$; combine Example 1.3(ii), Theorem 2, and the remarks in Section 1.2. □

3.4. It follows from Example 1.3(iii) that the coefficient of q^c in $F(\boldsymbol{m})=\sum_{\sigma\in C(\boldsymbol{m})} q^{r(\sigma)}$ equals the number of paths from $\boldsymbol{0}$ to $\boldsymbol{m}$ whose lesser index is c. We are interested in the number of restricted paths from $\boldsymbol{0}$ to $\boldsymbol{m}$ with lesser index c. In other words, the coefficient of q^c in

$$G_n(\boldsymbol{m})=\sum_{\sigma\in\bar{C}(\boldsymbol{m})} q^{r(\sigma)}$$

where $\bar{C}(\boldsymbol{m})$ is the set of paths from $\boldsymbol{0}$ to $\boldsymbol{m}$ restricted to $\bigcap_{i=1}^{n-1}\{m_i-m_{i+1}\geq 0\}$.

Theorem 8.

$$G_n(\boldsymbol{m})=q^{-n(n-1)(n-1)/6}\frac{(q)_{m_1+\cdots+m_n}}{(q)_{m_1+n-1}\cdots(q)_{m_{n-1}+1}(q)_{m_n}}\prod_{1\leq i<j\leq n}(q^{m_j+n-j}-q^{m_i+n-i}). \tag{3.2}$$

Proof. Apply Example 1.3(iii) to Theorem 2, then q-imitate the proof of Theorem 5. □

Remark. Formula (3.2) can be easily manipulated to give a q-analogy of the Frame–Robinson–Thrall formula in which form it strongly resembles the generating function for reverse plane partitions (Stanley [10, p. 270]). It would be interesting to find a direct combinatorial relationship between the two problems.

4. Plane partitions

A plane partition of a (e.g. Andrews [1, p. 179]) is an array whose sum is a:

$$a=\sum_{i,j\geq 0} a_{ij}, \quad \text{such that } a_{ij}\geq a_{i'j'} \quad \text{whenever } i\leq i', j\leq j'.$$

MacMahon [8, Section X] considered $F_n(p_1,\ldots,p_n)$, the generating function of plane partitions with $\leq n$ columns, unrestricted number of rows and $a_{1j}\leq p_j$ $(j=1,\ldots,n)$. $F_n(p_1,\ldots,p_n)$ also enumerates plane partitions of shape $p_1+p_2+\cdots+p_n$. Of course F_n is only defined for $p_1\geq\cdots\geq p_n$. MacMahon found a partial difference equation for $F_n(p_1,\ldots,p_n)$ [8, 220–221] which is easily obtainable by an inclusion–exclusion argument. Setting $\Delta_i=1-X_i^{-1}$, $i=1,\ldots,n$, the equation is

$$q^{p_1+\cdots+p_n}F_n=\Delta_1\cdots\Delta_n F \tag{4.1}$$

which holds in the "interior" of $\{p_1\geq p_2\geq\cdots\geq p_n\geq 0\}$, i.e., in $\{p_1>p_2>\cdots>$

$p_n > 0$). MacMahon's boundary conditions were

$$q^{p_1+\cdots+p_n}F_n = \begin{cases} \Delta_2 \cdots \Delta_n F_n & p_1 = p_2 \\ \cdot & \\ \cdot & \\ \cdot & \\ \Delta_1\Delta_2 \cdots \Delta_{i-1}\Delta_{i+1} \cdots \Delta_n F_n & p_i = p_{i+1} \\ \cdot & \\ \cdot & \\ \cdot & \\ \Delta_1 \cdots \Delta_{n-2}\Delta_n F_n & p_{n-1} = p_n \end{cases} \tag{4.2}$$

$$F_n(p_1, \ldots, p_{n-1}, 0) = F_{n-1}(p_1, \ldots, p_{n-1}),$$

again by inclusion–exclusion (e.g., on $p_1 = p_2$ it is not allowed to go to $(p_1 - 1, p_1, p_2, \ldots, p_n)$). MacMahon had a very hard time solving the above boundary value problem, and his solution was a rather messy determinant.

Our twist consists in extending F_n to $\{p_1 - p_2 \geq -1\} \cap \cdots \{p_{n-1} - p_n \geq -1\} \cap \{p_n \geq 0\}$ and requiring (4.1) to hold there. This forces (4.2) to become

$$\begin{aligned} &X_i^{-1}\Delta_1 \cdots \Delta_{i-1}\Delta_{i+1} \cdots \Delta_n F_n = 0, \quad p_i = p_{i-1}, \qquad i = 1, \ldots, n-1 \\ &F_n(p_1, \ldots, p_{n-1}, 0) = F_{n-1}(p_1, \ldots, p_{n-1}), \qquad p_n = 0, \end{aligned} \tag{4.2$'$}$$

or equivalently

$$\begin{aligned} &\Delta_2 \cdots \Delta_n F = 0, \qquad p_1 - p_2 = -1, \\ &\cdot \\ &\cdot \\ &\cdot \\ &\Delta_1 \cdots \Delta_{n-2}\Delta_n F = 0, \qquad p_{n-1} - p_n = -1, \\ &F_n(p_1, \ldots, p_{n-1}, 0) = F_{n-1}(p_1, \ldots, p_{n-1}). \end{aligned}$$

The partial difference Eq. (4.1) together with the boundary conditions (4.2″) uniquely define $F_n(p_1, \ldots, p_n)$ as a function in $\bigcap_{i=1}^{n-1} \{p_i - p_{i+1} \geq -1\} \cap \{p_n \geq 0\}$.

Theorem 9.

$$F_n(p_1, \ldots, p_n) = \prod_{1 \leq i < j \leq n} (1 - q^{j-i}X_iX_j^{-1})[1/(q)_{p_1} \cdots (q)_{p_n}] \tag{4.3}$$

From this follows immediately

Corollary 10. (MacMahon [8, p. 243]). *The generating function of all plane partitions, G_∞, is given by*

$$G_\infty = \prod_{k=1}^{\infty} (1 - q^k)^{-k}.$$

Proof of Theorem 9. We have to check that F_n as given by (4.3) satisfies (4.1) and the boundary conditions (4.2″). First note that each individual $X_iX_j^{-1}$ commutes with both sides of (4.1) thus so does $\prod(1-q^{j-i}X_iX_j^{-1})$. But $1/[(q)_{p_1}\cdots(q)_{p_n}]$ is a solution of (4.1) (Check!, *Hint:* $\Delta_i(1/(q)_{p_i})=q^{p_i}/(q)_{p_i}$) and therefore so is F_n. Now

$$\begin{aligned}\Delta_2\cdots\Delta_nF_n &= \Delta_2\cdots\Delta_n\prod(1-q^{j-i}X_iX_j^{-1})(q)_{p_1}^{-1}\cdots(q)_{p_n}^{-1}\\ &= \prod(1-q^{j-i}X_iX_j^{-1})\Delta_2\cdots\Delta_n(q)_{p_1}^{-1}\cdots(q)_{p_n}^{-1}\\ &= \prod(1-q^{j-i}X_iX_j^{-1})q^{p_2+\cdots+p_n}(q)_{p_1}^{-1}\cdots(q)_{p_n}^{-1}\\ &= q^{p_1+\cdots+p_n}\prod_{1\le i<j\le n}(1-q^{j-i}X_iX_j^{-1})q^{-p_1}(q)_{p_1}^{-1}\cdots(q)_{p_n}^{-1}\\ &= q^{p_2+\cdots+p_n}(1-X_1X_2^{-1})\,[\text{operator symmetric in } X_1, X_2]\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times(q)_{p_1}^{-1}\cdots(q)_{p_n}^{-1}\\ &= 0,\quad \text{on}\,\{p_1-p_2=-1\}\quad(\text{by Lemma 3}).\end{aligned}$$

The other conditions in (4.2″) are checked similarly. $F_n(p_1,\ldots,p_{n-1},0)=F_{n-1}(p_1,\ldots,p_{n-1})$ by the natural inductive hypothesis. □

5. Further applications

The same method applies to give the generating function of plane partitions with $\le r$ rows, $F_n(p_1,\ldots,p_n;r)$, namely

$$F_n(p_1,\ldots,p_n;r)=\prod_{1\le i<j\le n}(1-q^{j-i}X_iX_j^{-1}Y_i^{-1}Y_j)\prod_{i=1}^{n}\frac{(p_i+r_i)!}{(p_i)!\,(r_i)!}\tag{5.1}$$

evaluated at $r_1=\cdots=r_n=r$. Here $Y_i^{-1}f(r_i)=f(r_i-1)$, $i=1,\ldots,n$; $(a)!=(q)_a$. It is possible to obtain MacMahon's determinant formula by transforming the product of operators featuring in (4.3) and (5.1) to a determinant (using Vandermode's determinant) but this is rather pointless ever since Gessel came out with his very general and elegant paper [4]. It is also possible to milk (5.1) to obtain $A(n,k;r)=F_n(k,k,\ldots,k;r)$ without the intervention of determinants.

Formula (4.3) makes clear the relationship, first noticed by MacMahon, between plane partitions and lattice permutation (alias the Ballot problem). Putting $q=1$ in (4.3) yields

$$F_n(p_1,\ldots,p_n)=\prod_{1\le i<j\le n}(1-X_iX_j^{-1})[p_1!\cdots p_n!]^{-1}=f_p/(p_1+\cdots+p_n)!.$$

6. Pseudo plane partitions

Definition. A composition of λ, $\lambda=\sum_{i=1}^{n}\lambda_1$ is a k-pseudo partition if $\lambda_i-\lambda_{i+1}\ge -k$. (0-pseudo partition = "partition"; (-1)-pseudo partition = "strict partition").

Theorem 12. *Let $F_n(p_1, \ldots, p_n)$ be the generating function of n-column arrays $a = \sum a_{ij}$, where $a_{1j} \leqslant p_j$ $(j = 1, \ldots, n)$, s.t. $a_{i,j} - a_{i,j+1} \geqslant -(k-1)$ and $a_{i,j} - a_{i+1,j} \geqslant 0$, (for $k = 1, 2, 3, \ldots$). This means that the columns are regular partitions, while the rows are $(k-1)$ pseudo partitions. We have*

$$F_n(p_1, \ldots, p_n) = \prod_{1 \leqslant i < j \leqslant n} [1 - (q^{j-i} X_i X_j^{-1})^k] (q)_{p_1}^{-1} \cdots (q)_{p_n}^{-1}.$$

Proof. $F_n(p_1, \ldots, p_n)$ satisfies Eq. (4.2) under the boundary conditions

$$\begin{array}{ll} \Delta_2 \cdots \Delta_n F_n = 0, & p_1 - p_2 = -k, \\ \cdot & \\ \cdot & \\ \cdot & \\ \Delta_1 \cdots \Delta_{n-1} F_n = 0, & p_{n-1} - p_n = -k, \\ F_n = F_{n-1}, & p_n = 0. \end{array} \tag{6.1}$$

The proof is similar to the proof of Theorem 9, this time using the second half of Lemma 3.

Corollary 13. *Let G_n be the generating function enumerating arrays discussed in Theorem 12, with $\leqslant n$ columns, i.e., $G_n = F_n(\infty, \ldots, \infty)$. Then,*

$$G_n (q^k)_{n-1} (q^k)_{n-2} \cdots (q^k)_1 / (q)_\infty^n,$$

where

$$(q^k)_i = (1-q^k)(1-q^{2k}) \cdots (1-q^{ik}).$$

In the same vein,

Theorem 14. *Let $F_n(p_1, \ldots, p_n)$ be the generating function of n column arrays, $a = \sum a_{ij}$, whose columns are strict partitions and whose rows are 1-pseudo partitions:*

$$a_{i,j} - a_{i,j+1} \geqslant -1, \quad a_{i,j} - a_{i+1,j} \geqslant 1, \quad a_{1j} \leqslant p_j \qquad (j = 1, \ldots, n).$$

F_n is given by

$$F_n(p_1, \ldots, p_n) = \prod_{1 \leqslant i < j \leqslant n} (1 - q^{j-i} X_i X_j^{-1}) l(p_1) \cdots l(p_n)$$

where

$$l(p) = (1+q)(1+q^2) \cdots (1+q^p).$$

Corollary 15. *Let $G_n = F_n(\infty, \ldots, \infty)$, then*

$$G_n = (q)_{n-1} (q)_{n-2} \cdots (q)_1 \prod_{i=1}^{\infty} (1+q^i)^n.$$

As a closing remark let us mention that it is possible to obtain generating functions for the above entities for which the number of rows is restricted. These formulas are similar to (5.1).

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