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NOTE

# GARSIA AND MILNE'S BIJECTIVE PROOF OF THE INCLUSION–EXCLUSION PRINCIPLE

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Although the following proof is implicit in Garsia and Milne's paper [1], it is so elegant that we felt that it should be presented by itself for the benefit of the general mathematical public. The idea behind the proof was further exploited by Remmel [2] and Wilf [3].

Consider a set $A$ of elements each of which possess a (possibly empty) subset of the properties $\{1, \ldots, n\}$. The inclusion–exclusion principle states that the number of elements with no properties is

$$\sum_{I \subset \{1,\ldots,n\}} (-1)^{|I|} |A_I|. \tag{1}$$

Here, for any subset $I$ of $\{1, \ldots, n\}$, $A_I$ denotes the set of elements having all the properties of $I$ and, for any set $B$, $|B|$ denotes the number of elements of $B$.

Our proof starts by introducing the much larger set $\mathcal{A}$ of all possible pairs $(a, J)$ where $a$ is an element of $A$ and $J$ is a subset of the set of properties of $a$. The pair $(a, J)$ is *even* or *odd* according to whether $|J|$ is even or odd respectively. We next observe that for a fixed $I \subset \{1, \ldots, n\}$, $(a, I)$ is a legitimate pair if and only if $a \in A_I$. It follows that (1) expresses the difference between the number of even and odd pairs.

For any $a \in A$ let $s(a)$ be its smallest property. Define the following mapping from $\mathcal{A}$ to itself:

$$T(a, J) = \begin{cases} (a, J \cup s(a)), & s(a) \notin J, \\ (a, J / s(a)), & s(a) \in J. \end{cases}$$

This is a parity changing involution which is defined everywhere *except* on pairs of the form $(a, \emptyset)$ where $a$ is a property-less element of $A$. It follows that the odd pairs of $\mathcal{A}$ are equinumerous with the even pairs of $\mathcal{A}$ which are *not* of the above form. This implies (1) since the pairs $(a, \emptyset)$ for which $a$ is property-less are obviously equinumerous with the set of property-less elements of $A$.

Our proof readily extends to prove the following generalized version of the inclusion–exclusion principle. Let $t_1, \ldots, t_n$ be commuting indeterminates and for $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ denote $t_I = t_{i_1} \cdots t_{i_k}$ and $(t-1)_I = (t_{i_1} - 1) \cdots (t_{i_k} - 1)$, then if for $a \in A$, Prop$(a)$ denotes the set of properties of $a$ then

$$\sum_{a \in A} t_{\text{prop}(a)} = \sum_{I \subset \{1,\ldots,n\}} |A_I| \, (t-1)_I.$$

## References

[1] A.M. Garsia and S.C. Milne, A Rogers–Ramanujan bijection, J. Combin. Theory Ser. A 31 (1981) 289–339.
[2] J. Remmel, Bijective proofs of some classical partition identities, to appear.
[3] H.S. Wilf, Sieve equivalence in generalized partition theory, J. Combin. Theory Ser. A 34 (1983) 80–89.