Inequalities are deep, while equalities are shallow. Nevertheless, it sometimes happens that a deep inequality, \( \mathbf{A} \), follows from a mere equality \( \mathbf{B} \), which, in turn, follows from a more general, and trivial\(^2\) identity \( \mathbf{C} \).

In this note we demonstrate this, following [3], with \( \mathbf{A} := \) Bombieri’s norm inequality [2]\(^3\), \( \mathbf{B} := \) an identity of Reznick [5], and \( \mathbf{C} := \) an identity of Beauzamy and Dégot [3]. This exposition differs from the original only in the punch line: I give a 1-line proof of \( \mathbf{C} \), using Chu’s identity.

Let \( P(x_1, \ldots, x_n) \) and \( Q(x_1, \ldots, x_n) \) be two polynomials in \( n \) variables:

\[
P = \sum_{i_1, \ldots, i_n \geq 0} a_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n}, \quad Q = \sum_{i_1, \ldots, i_n \geq 0} b_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n}.
\]

The Bombieri inner product [2] is defined by

\[
[P, Q] := \sum_{i_1, \ldots, i_n \geq 0} (i_1! \cdots i_n!) \cdot a_{i_1, \ldots, i_n} b_{i_1, \ldots, i_n},
\]

and the Bombieri norm, \( \|P\| := \sqrt{[P, P]} \).

**Bombieri’s Inequality.** Let \( P \) and \( Q \) be any homogeneous polynomials in \( (x_1, \ldots, x_n) \), then

\[
\|PQ\| \geq \|P\| \|Q\|.
\]

\(^1\)This note was written while the author was on leave (Fall 1993) at the Institute for Advanced Study, Princeton. I would like to thank Don Knuth for a helpful suggestion.

\(^2\)Trivial to verify, not to conceive!

\(^3\)It was needed by Beauzamy and Enflo in their research on deep questions on Banach spaces. It also turned out to have far reaching applications to computer algebra! [1].
In order to state $B$ and $C$, we need to introduce the following notation. $D_i := ∂/∂x_i$, $(i = 1, ..., n)$, $P^{(i_1, ..., i_n)} := D_1^{i_1} ... D_n^{i_n} P$, and for any polynomial $A(x_1, ..., x_n)$, $A(D_1, ..., D_n)$ denotes the linear partial differential operator with constant coefficients obtained by replacing $x_i$ by $D_i$.

A follows almost immediately from ([5][3]):

Reznick's Identity B. For any polynomials $P, Q$ in $n$ variables:

$$\|PQ\|^2 = \sum_{i_1, ..., i_n ⩾ 0} \|P^{(i_1, ..., i_n)}(D_1, ..., D_n)Q(x_1, ..., x_n)\|^2_{i_1! ... i_n!}.$$  

Beauzamy and Dégot's Identity C. For any polynomials $P, Q, R, S$ in $n$ variables:

$$[PQ, RS] = \sum_{i_1, ..., i_n ⩾ 0} \frac{[R^{(i_1, ..., i_n)}(D_1, ..., D_n)Q(x_1, ..., x_n), P^{(i_1, ..., i_n)}(D_1, ..., D_n)S(x_1, ..., x_n)]}{(i_1! ... i_n!)}.$$  

Proof of B $⇒$ A. Pick the terms for which $i_1 + ... + i_n$ equals the (total) degree of $P$, let’s call it $p$, and note that for those $(i_1, ..., i_n)$, $P^{(i_1, ..., i_n)}(x_1, ..., x_n) = (i_1! ... i_n!)a_{i_1, ..., i_n}$, so

$$\sum_{i_1 + ... + i_n = p} \|P^{(i_1, ..., i_n)}(D_1, ..., D_n)Q(x_1, ..., x_n)\|^2_{i_1! ... i_n!} = \sum_{i_1 + ... + i_n = p} \|a_{i_1, ..., i_n}Q(x_1, ..., x_n)\|^2 \cdot (i_1! ... i_n!)$$  

$$= \left(\sum_{i_1 + ... + i_n = p} (a_{i_1, ..., i_n})^2 \cdot (i_1! ... i_n!))^2 \|Q(x_1, ..., x_n)\|^2 = \|P\|^2 \|Q\|^2.$$  

Proof of C $⇒$ B. Take $R = P$ and $S = Q$.

Proof of C. Both sides are linear in $P$, in $Q$, in $R$, and in $S$, so it suffices to take them all to be typical monomials, $(P = x_1^{p_1} ... x_n^{p_n}$, and similarly for $Q, R, S$), for which the assertion follows immediately by applying Chu’s [4] identity  

$$\sum_{i⩾0} \binom{r}{i} \binom{s}{p-i} = \binom{r+s}{p},$$  

to $r = r_t$, $s = s_t$, $p = p_t$, $(t = 1 ... n)$, (using $i_t$ for $i$), and taking their product. Q.E.D.

REFERENCES


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4Rediscovered in the 18th century by Vandermonde. Proved by counting, in two different ways, the number of ways of picking $p$ lucky winners out of a set of $r$ boys and $s$ girls.

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Answer on page 910.