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A UNIFIED APPROACH TO MACDONALD'S ROOT-SYSTEM CONJECTURES*

DORON ZEILBERGER†

Dedicated to Dennis Stanton and John Stembridge for reminding me that antisymmetry is even more powerful than symmetry.

"*Yes, of course. It works with herring, but will it work with ferrous metals?*"
(*Woody Allen* [Al]).

Abstract. Using ideas of Stembridge and Stanton a method is presented that should settle the Macdonald (and the more refined Macdonald–Morris) root-system conjectures for any *specific* root system, provided there is sufficient computer time, memory space, and (for now) some luck. The method consists of an algorithm that reduces Macdonald's conjecture for a given root system to a finite, albeit long, algebraic calculation, which is then performed using computer algebra. The method is illustrated by proving the so far open $G_2^\vee$ case of the Macdonald–Morris conjectures. The question that remains is: will it work with E_8 (and F_4, E_6, E_7)?

Key words. Macdonald's root-system conjectures, constant term, q-analogue, computer algebra

AMS(MOS) subject classifications. 05A15, 05A17, 33A15, 33A75

Introduction. This paper is about Macdonald's *root-system* conjectures. In order to understand it, it is necessary to know a little bit about root systems and their Weyl groups. While it seems obvious that before one can talk about *root-system* conjectures one has to know about root systems, this is not the case for many of the papers on this subject. By the classification theorem for root systems, it is possible to spell out what the conjectures say for each of the four infinite families and the five exceptional root systems, and then treat each case separately [Mo]. Although only one root-system is treated at a time in this paper, its method is cast in the general root-system mold.

Historically root systems first came up in the deep and sophisticated theory of Lie algebras. This noble birth gave them a fancy aura that scared away many a plebeian mathematician. However, root systems are really very simple-minded, combinatorial-geometrical structures and it is possible, perhaps even preferable, to study root systems without knowing anything about Lie algebras.

A root system is a finite collection of vectors, called roots, in regular (Euclidean) space such that if you place a mirror perpendicular to any of them, the image of the visible part that is reflected in the mirror coincides exactly with the invisible part behind the mirror. Furthermore, the vector difference between any root and its image under any such mirror is an *integer* multiple of the root corresponding to the mirror (i.e., the root that is perpendicular to the mirror). These two conditions are very strong and it turns out (the classification theorem) that all irreducible root systems fall into five infinite families and five exceptionals. If you add the condition that these vectors can only be parallel to their negatives (reduced root systems) then one infinite family (BC_n) drops out.

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† Department of Mathematics, Drexel University, Philadelphia, Pennsylvania 19104.

An excellent treatment of root systems and Weyl groups is given in Chapters 2 and 10 of Carter's book [C]. These two chapters are completely independent of the rest of the book and are entirely elementary. This paper can be understood by any one who has read the first two sections of Chapter 2 and the first two sections of Chapter 10 of [C]. A comprehensive and (surprisingly) quite readable account is given in [Bo], but for the present paper [C] is more than enough.

Notation. The Macdonald conjectures are about certain multivariable Laurent polynomials. A Laurent polynomial is a linear combination of monomials that may have negative integer exponents as well as positive integer exponents. For example $x+1+x^{-1}$ is a Laurent polynomial in one variable and $x+y+x^{-1}y^2$ is one in two variables. Usually x denotes a vector of variables, $x=(x_1,\cdots,x_l)$ and α a vector of integers, $\alpha=(\alpha_1,\cdots,\alpha_l)$. Also

$$x^\alpha = x_1^{\alpha_1}\cdots x_l^{\alpha_l}.$$

For example $x^{(1,-2,5)}=x_1x_2^{-2}x_3^5$.

For the roots α of a root system, x^α, are often called "formal exponentials." But since all root systems can be made to have all their roots with integer components, these exponentials can be easily defrocked of their formality. The root lattice of a root system consists of all integer linear combination of roots, and all our Laurent polynomials will be linear combinations of monomials x^γ for γ in the root lattice. The Weyl group W of a root system [C, Chap. 2] acts on the roots, and by linearity on the root lattice. The elements w of the Weyl group W are made to act on monomials by

$$w(x^\gamma)=x^{w(\gamma)}$$

and by linearity on all Laurent polynomials. For example, if $w(\alpha_1,\alpha_2)=(-\alpha_2,\alpha_1)$, then

$$w(x^{-1}y^2+3+x^5y^{-2})=x^{-2}y^{-1}+3+x^2y^5.$$

A Laurent polynomial G is *symmetric* with respect to the Weyl group W if $w(G)=G$ for every w in W. The *sign* of an element w of W, written sgn (w), may be defined as [C, p. 18] $(-1)^{n(w)}$, where $n(w)$ is the number of positive roots that w turns into negative roots, i.e., the number of elements in the set $w(R^+)\cap R^-$. A Laurent polynomial G is *antisymmetric* if for any w in the Weyl group W, $w(G)=\text{sgn}(w)G$.

C.T. stands for "the constant term of" (in $x=(x_1,\cdots,x_l)$), and $|A|$ denotes the number of elements of the finite set A. The letter l usually denotes the rank of R, and $d_1,\cdots,d_l$, are the "fundamental invariants" [C, p. 155] of R.

The $(\ )_a$ q-notation will be used extensively. $(y;Q)_a$, the q-analogue of $(1-y)^a$ to base Q, is defined by

$$(y;Q)_a=(1-y)(1-Qy)(1-Q^2y)\cdots(1-Q^{a-1}y),$$

and whenever the "base" Q happens to be q we will omit it: $(y)_a=(y;q)_a$. The standard base of Euclidean space is denoted by $\{e_i\}$, $e_i=(0,0,\cdots,0,1,0,0,\cdots,0)$, where all the components are zero except the i component that is 1. Of course $x^{e_i}=x_i$.

1. Conjectures. In 1962, in his study of the statistical theory of complex systems, Dyson [D1] conjectured

$$\text{(D)}\qquad \text{constant term of } \prod_{1\leqq i\neq j\leqq n}\left(1-\frac{x_i}{x_j}\right)^a=\frac{(na)!}{a!^n}.$$

His conjecture was soon proved by Gunson [Gu] and Wilson [W] and Good [Goo] gave a beautiful proof some years later.

When Macdonald saw Dyson's conjecture (D) he saw the root system A_{n-1}. Indeed, since

$$A_{n-1} = \{e_i - e_j;\ 1 \leqq i \neq j \leqq n\} \quad \text{and} \quad x^{e_i - e_j} = \frac{x_i}{x_j},$$

(D) can be written as

$$\text{constant term of} \prod_{\alpha \in A_{n-1}} (1 - x^\alpha)^a = \frac{(na)!}{a!^n}.$$

He then wondered what happens if A_{n-1} is replaced by other root systems.

The case $a = 1$ of Dyson's conjecture (D) is an almost immediate consequence of the Vandermonde determinant identity and the constant term then is $n! = n(n-1) \cdots (2)$. Now the Vandermonde determinant identity has a celebrated root-system analogue: Weyl's denominator identity (e.g., [C, p. 149]), and imitating the argument that proved (D) for $a = 1$ yields, for any root-system R with Weyl group W,

$$\text{C.T.} \prod_{\alpha \in R} (1 - x^\alpha) = |W|.$$

For $R = A_{n-1}$, $W = S_n$ and since $|W| = |S_n| = n!$, this agrees with the $a = 1$ case of (D).

So the $a = 1$ case of (D) has a nice root-system analogue. What about general a? It is well known that $|W|$ factorizes nicely [C, 9.3.4(i), p. 133]:

$$|W| = d_1 d_2 \cdots d_l,$$

where $d_1, \cdots, d_l$ are the "fundamental invariants" of the Weyl group W (these fundamental invariants are, among other things, the degrees of the generators of the algebra of polynomials invariant under W). For A_{n-1} these invariants are $2, 3, \cdots, n$ (the degrees of the elementary symmetric functions!). Rewriting the right-hand side of (D) as

$$\binom{2a}{a} \cdots \binom{na}{a},$$

Macdonald [Ma3] conjectured that

$$\text{(M)} \qquad \text{constant term of} \prod_{\alpha \in R} (1 - x^\alpha)^a = \binom{d_1 a}{a} \cdots \binom{d_l a}{a}.$$

Macdonald was also able to prove the special case $a = 2$, and by using Selberg's integral [Se] he proved the B_n, C_n, and D_n cases. Recently Habsieger [Hab1] and Zeilberger [Z2] proved the G_2 case. For $R = F_4$, E_6, E_7, and E_8, (M) is still open, as far as I know.

Next Macdonald went on to formulate a "q-analogue" of (M). Andrews ([An1]; see also [An2]) already formulated a q-analogue of (D) in 1975. Actually Andrews conjectured a q-analogue of a more general conjecture of Dyson, and his conjecture specializes to the following q-analogue of (D):

$$\text{(qD)} \qquad \text{C.T.} \prod_{1 \leqq i < j \leqq n} \left(\frac{x_i}{x_j}\right)_k \left(\frac{q x_j}{x_i}\right)_k = \frac{[nk]!}{[k]!^n} \left(= \begin{bmatrix} 2k \\ k \end{bmatrix} \cdots \begin{bmatrix} nk \\ k \end{bmatrix} \right).$$

The general Andrews conjecture was proved in [Z-B].

Motivated by this and (M) Macdonald [Ma3] conjectured

$$\text{(qM)} \qquad \text{C.T.} \prod_{\alpha \in R^+} (x^\alpha)_k (q x^{-\alpha})_k = \begin{bmatrix} d_1 k \\ k \end{bmatrix} \cdots \begin{bmatrix} d_l k \\ k \end{bmatrix}.$$

Macdonald [Ma3] was able to prove (qM) for $k=1,2$ and $k=\infty$. For $k=\infty$ (qM) is a consequence of his own famous Macdonald Weyl identities [Ma2] (many special cases of which were known to Dyson [D2], but Dyson "missed the opportunity" to see the connection to root systems). For general k (qM) is only known to date for $R=A_n$ [Z-B] and G_2 ([Hab1], [Z2]). Hanlon [Han1] did the limiting case $n=\infty$ of B_n, C_n, and D_n.

One of the greatest delights of mathematics is the interplay between the abstract and the concrete, the general and the special. Whenever one has a general result or conjecture, it is very instructive to see what it says in special cases, and studying these special cases often sheds new light on the general case. Morris [Mo] took Macdonald's conjectures and made them explicit for all the root systems. Then by studying the G_2 case and playing with MACSYMA he was able to come up with a more general G_2-Macdonald conjecture, involving two parameters a and b instead of the single parameter k:

$$\text{C.T.} \prod_{\alpha\in \text{short } G_2} (1-x^{\alpha})^{a} \prod_{\alpha\in \text{long } G_2} (1-x^{\alpha})^{b} = \frac{(3a+3b)!(3b)!(2a)!(2b)!}{(2a+3b)!(a+2b)!(a+b)!a!b!b!}.$$

This was encouraging because it always helps to have more parameters (recall Polya's dictum: "the more general the easier"). Indeed Good's ([Goo]; see also [An2], [As3]) elegant proof of Dyson's conjecture (D) proceeds by proving the more general formula (also conjectured by Dyson [D1]):

$$\text{(D}'\text{)} \qquad \text{C.T.} \prod_{1\leq i\neq j\leq n} \left(1-\frac{x_i}{x_j}\right)^{a_i} = \frac{(a_1+\cdots+a_n)!}{a_1!\cdots a_n!},$$

and the extra elbow room provided by the n parameters $a_1,\cdots,a_n$ is crucial.

Morris sent his G_2 conjecture to Macdonald and, once again, Macdonald saw the right root-system generalization ([Ma3], [Mo]). Now there is a parameter associated with each root length. (Since A_l, D_l, E_6, E_7, E_8 have only one root length the generalization is void for them. For B_l, C_l, G_2, F_4 we have two parameters and BC_l has three parameters.)

Macdonald soon found a q-analogue [Ma3, 3.1]: if k_α are nonnegative integers such that $k_\alpha=k_\beta$ if α and β have the same length, then

$$\text{(qM-M1)} \qquad \text{C.T.} \prod_{\alpha\in R^+} (x^{\alpha})_{k_\alpha}(qx^{-\alpha})_{k_\alpha} = \text{a certain explicit product.}$$

I already mentioned that the case $k=\infty$ of (qM) is a consequence of Macdonald's Weyl identitities [Ma1]. These identities are the analogue of the Weyl denominator formula for affine root systems. (Incidentally these were "the tip of the iceberg" that motivated the representation theory of Kac-Moody algebras [Kac, p. xiii], but that's another story.) It turned out that the Macdonald-Morris conjectures (qM-M1) can be viewed as the "truncated form" of Macdonald's identities for the so-called $S(R)$ affine root systems ([Ma3, p. 999]; see [Ma1] and [Mo] for definitions of affine root systems). The classification theory of affine root systems [Ma1] says that the irreducible ones are either of the form $S(R)$ or $S(R)^{\vee}$. It was thus natural for Macdonald to formulate his conjectures as the truncated form of his identities and that led to the ultimate generalization ([Ma3, Conjecture 3.3], [Mo, pp. 25, 26]): Let k_α be as before and let u_α be certain *constant* integers (depending on the affine root system) that satisfy $u_\alpha=u_\beta$ whenever α and β have the same length (see [Ma1] or [Mo] for their values, for example for $S(G_2)^{\vee}$, $u_{\text{short}}=1$ and $u_{\text{long}}=3$). Let R be the underlying finite root system;

then,

(qM-M2) C.T. $\prod_{\alpha \in R^+} (x^\alpha; q^{u_\alpha})_{k_\alpha} (q^{u_\alpha} x^{-\alpha}; q^{u_\alpha})_{k_\alpha}$ = a certain explicit product.

The Macdonald conjectures, like most interesting mathematics, lie on the crossroads of several subjects, and so appeal to a wide spectrum of mathematicians. Lie algebraists suspect that, like the Macdonald identities, they are the tip of a deep algebraic iceberg [Han1]-[Han3], [Stanl1], [Stanl2]. Analysts [Mo], [As1]-[As3] see many interesting examples of multivariate hypergeometric series identities, "a topic about which little is currently known" [Mo, p. 4]. Geometers wonder whether there are things about root systems that they do not know, and combinatorialists [Z-B], [Br1], [Br2], [C-H], [B-G] are challenged to develop a combinatorial theory of Weyl groups that will emulate the rich theory of the symmetric group.

But regardless of our parochial interests and prejudices, we are all awed by the simplicity of these conjectures. The statement of the Macdonald conjectures, for any specific root system, can be explained to a high school student, but the proofs elude us.

2. Approaches. I will now give a very brief survey of the various approaches that have been used to tackle the Macdonald conjectures.

Selberg's integral. This fascinating generalization of Euler's beta integral was discovered by Selberg [Se] in 1944 but lay dormant for about 35 years, partially because it was ahead of its time, partially because it was written in Norwegian and partially because Selberg wrote it before he got *really* famous. This sleeping beauty awoke from its deep slumber when Enrico Bombieri consulted Selberg about a certain conjectured definite integral of Mehta [Me], [As3], [Ma3] and Selberg dug his old paper out of his files. It turned out that Mehta's conjecture (that has been open for about 15 years) is an easy consequence, via a limiting process, of Selberg's integral.

Mehta's conjecture [Me], which can be thought of as an integral analogue of Dyson's conjecture (D), also received root-system analogues by Macdonald [Ma3, § 5]. Beckner and Regev (see [Ma3, § 5]) showed how Selberg's integral can be used to get these root-system-Mehta conjectures for the classical root systems.

Macdonald [Ma3] showed, by a clever change of variable, that Selberg's integral is equivalent to the BC_n, $q = 1$, case of (qM-M), which implies (M) for B_n, C_n, and D_n. Using a corollary of Selberg's integral, due to Morris [Mo, p. 94], Zeilberger [Z2], and Habsieger [Hab1] proved the G_2 case of (M). Aomoto [Ao] has recently found a very ingenious proof of Selberg's integral by using integration by parts, recurrences, and symmetry; see [As3] for a nice account.

By employing Jackson's q-analogue of integration, Askey [As1] formulated an elegant q-analogue of Selberg's integral that has recently been proved by Kadell [Kad1] and by Habsieger [Hab2]. Kadell q-analogized Aomoto's proof and Habsieger used Selberg's original method coupled with some brilliant ideas of his own. Kadell and Habsieger also showed that their Askey- q-Selberg identity implies the q-analogue of Morris' identity mentioned above. This q-analogue, conjecture by Morris himself [Mo], enabled Habsieger [Hab1] and Zeilberger [Z2] to prove the G_2 case of (qM-M1). Incidentally, Kadell and Habsieger's q-Morris identity contains, as a special case, the A_n case of (qM-M) (first proved in [Z-B]).

Counting tournaments. I already mentioned that the case $a = 1$ of (D) follows from the Vandermonde determinant identity. The case $n = 3$ is also classical, being equivalent to Dixon's identity [An1]. Both these classical identities received beautiful combinatorial proofs. Gessel [Ge] (see also [An2, 4.4]) gave an elegant graph-theoretical proof of Vandermonde's determinant identity by counting tournaments, and

Foata [F], [C-F] gave a gorgeous proof of Dixon's identity by using multitournaments on three players.

Combining these two pretty proofs, Zeilberger [Z1] managed to give a purely combinatorial proof of Dyson's conjecture (D′) (and thus of (D)). In that paper Zeilberger wrote: "We believe that our proof has a good chance of being generalized, because most combinatorial proofs involving binomial coefficients have q-analogues. However, another idea is still needed since the obvious q-generalization fails." The "obvious generalization" was to q-count words by using either the number of inversions or the major index as the "statistics" because both yield the q-multinomial coefficients. But neither of these worked. The new idea that was needed was to introduce a brand-new statistic, the z-index, and to prove that it, too, yields the q-multinomial coefficients. This was done in [Z-B], which contains a proof of Andrews' conjecture (and hence of the A_{n-1} case of (qM)).

Motivated by the success of the combinatorial method, there were attempts to extend it to general root systems [Br1], [Br2], [C-H]. Although these papers contain some very promising ideas, they failed, so far, even to prove the G_2 case. I should also mention [B-G], that, using the methods of [Z-B], contains interesting extensions of Andrews' conjecture, and [Gr], that gives an elegant MacMahon-style combinatorial proof for the above-mentioned fact that the z-statistics yield the q-multinomial coefficients.

Lie algebra cohomology. Hanlon [Han2], [Han3] found an interesting formulation and refinement of Macdonald's conjectures in the context of the cyclic homology of the exterior product of a Lie algebra with $\mathbb{C}[t, t^{-1}]$. Besides the considerable intrinsic merit of this approach, it also serves to make the conjectures accessible and appetitizing to all those sophisticates who are unwilling or unable to think in terms of the simple formulation of the original conjectures.

Hypergeometric $SU(N)$. Milne [Mi] found an elegant elementary proof of the $A_l^{(1)}$ case of Macdonald's identities. It is very possible that Milne's deep generalized hypergeometric theory will, one day, contain the Macdonald conjectures as a very special case.

Schur functions. Stanley [Stanl1], [Stanl2] found an interesting connection between the A_n cases and Schur functions. This connection was further explored by Stembridge [Ste1], [Ste2] and Goulden [Gou]. While these works do not try to prove the Macdonald conjecture per se, they Schur do give lots of insight. Indeed, it was exactly this study that led Stembridge [Ste3] to his elegant proof discussed below. It is not unlikely that a similar study of characters of general simple Lie algebras will lead us in the right direction.

I should also mention the interesting character sums analogues of Evans [E] and the fascinating connection between Mehta type integrals, PI rings and the representation of the symmetric group found by Regev [R1], [R2], and further explored by Cohen and Regev [C-R].

In April 1986, Dennis Stanton told me that John Stembridge had a short and elementary proof of the A_{n-1} case of (qM) (or equivalently, the equal parameter case of Andrews' q-Dyson conjecture). At first I was only mildly interested, since Kadell and Habsieger had just then completed, independently, the proof of Askey's conjectured q-Selberg integral ([Hab2], [Kad1], mentioned above) and also showed that it implies the q-Morris conjecture, that in turn implies the A_{n-1} case of (qM). In fact, I saw [Z3] how to use the Aomoto-Kadell method to get the q-Morris directly, without q-Selberg.

I wrote to John Stembridge anyway, requesting an account of the proof, and received from him a barely legible xerox copy of a three-page handwritten sketch that

Dennis Stanton had prepared. When I finally understood the proof I got excited. At long last a proof that has a "root-systemy" flavor! Although the proof was only of the A_{n-1} case, it had some universal root-system elements in it, and it used properties of the symmetric group that pass verbatim to any Weyl group, that I was sure that it should extend to the general case.

This was, of course, also realized by John Stembridge himself as he pointed out later when he finally got around to writing the paper [Ste3]. However, his proof took advantage of a certain "miracle" that seemed to occur only for A_{n-1}. Surely what was needed was to get rid of the dependence on the miracle, possibly by sacrificing elegance. I thought about that all through the summer (while taking care of my newborn daughter Tamar) and the result is this paper. (The fall was spent programming the algorithm and debugging the programs. If nothing else, this project made me a fairly competent C programmer.)

As John Stembridge told me himself, his proof, as well as parts of his impressive thesis [Ste1], were largely motivated and inspired by Dennis Stanton's ingenious proof of Macdonald's Weyl denominator identity for the classical root-systems [Stant1], [Stant2].

Using these beautiful ideas of Stembridge and Stanton, I will present a method that systematically handles the Macdonald conjectures for any given, fixed, root system, provided there are sufficient computer resources, and, for the time being, some luck. What I do know for sure is that it works for the (already known) A_2 and G_2 cases and for the (so far open) $G_2^\vee$ case (§ 9). Besides, I am almost sure that the element of luck can be disposed of and that the method can be proved to constitute an effective algorithm for settling the Macdonald and the Macdonald-Morris conjectures for any given root system. Of course, that by itself would not constitute a proof, or even an effective algorithm for the general conjecture, because there are an infinite number of root systems.

On the other hand, it is very possible that the A–D cases of the Macdonald and Macdonald-Morris conjectures will soon be settled by either using the Askey q-Selberg integral [Kad1], [Hab2] directly, or by using similar methods of proof. In that case we will only be left with the seven exceptional cases (G_2 and its dual, F_4 and its dual, and E_6, E_7, and E_8, but since the first two are already known this leaves us with five cases). These should succumb to the method of this paper (at least in principle, and barring very bad luck). But even if that would turn out to be the case, it would certainly not be the proof from *the book*. The ultimate proof should be "classification-free" and take care of all root systems at once.

To give a very apt analogy, the Weyl denominator formula [C, p. 149] can be proved case by case. A_{n-1} is just the Vandermonde determinant identity, which is an elementary exercise in determinants. The cases B_n, C_n, and D_n also specialize to simple algebraic identities that can be easily proved by induction. The remaining exceptional cases, G_2–E_8, give rise to finite polynomial identities that can be checked by computer (although I have to admit that, for E_8, even the CRAY will take "a while" to handle the 2^{120} terms). However, there is a beautiful "classification-free" proof of Weyl's identity that can be found in Carter's book ([C, § 10.1]).

I believe that besides the instant gratification that the present method brings, it is also an important step toward the ultimate proof. Unlike any previous approach, it makes use of the general root-system–Weyl group framework, and thus may pave the way to the final proof. In addition, it also provides a "laboratory" for computing other coefficients, besides the constant term, for any specific root system (see below). This may lead us to formulate a yet more general conjecture, and this more general conjecture may very well turn out to be much easier to prove.

3. Antisymmetry.

No two fermions can exist in identical quantum states.
(*Wolfgang Pauli*)

Let R be a root system and let us define

$$F'_k(x) = \prod_{\alpha \in R^+} (x^\alpha)_k (qx^{-\alpha})_k, \qquad H'_k = \text{C.T. } F'_k(x). \tag{3.1}$$

Macdonald's conjecture (qM) asserts that H'_k has a nice explicit form (namely, the right-hand side of (qM)). In any case, whether (qM) is true or false, our goal will be to compute H'_k. It turns out (and this observation is due to Macdonald, although Stembridge was the one to realize its full significance) that one can consider instead

$$F_k(x) = \prod_{\alpha \in R^+} (x^\alpha)_k (qx^{-\alpha})_{k-1} \tag{3.2}$$

and

$$H_k = \text{C.T. } F_k(x).$$

This is so because of the fact, soon to proved, that H_k and H'_k are related by a simple formula

$$H'_k = H_k \prod_{i=1}^{l} \left(\frac{1-q^{kd_i}}{1-q^k} \right). \tag{3.3}$$

The reason why it is better to consider the constant term of F_k rather than that of F'_k is that F_k is a much nicer Laurent polynomial: it is almost antisymmetric.

Indeed, by peeling off the first layer out of the $(x^\alpha)_k$ in (3.1) we get (since $(y)_k = (1-y)(qy)_{k-1}$)

$$\begin{aligned} F_k(x) &= \prod_{\alpha \in R^+} (1-x^\alpha) \prod_{\alpha \in R^+} (qx^\alpha)_{k-1} (qx^{-\alpha})_{k-1} \\ &= \prod_{\alpha \in R^+} x^{\alpha/2} (x^{-\alpha/2} - x^{\alpha/2}) \prod_{\alpha \in R} (qx^\alpha)_{k-1} \\ &= x^\delta \prod_{\alpha \in R^+} (x^{-\alpha/2} - x^{\alpha/2}) \prod_{\alpha \in R} (qx^\alpha)_{k-1} \end{aligned} \tag{3.4}$$

(δ is one half the sum of all the positive roots). Let

$$G_k(x) = x^{-\delta} F_k(x). \tag{3.5}$$

Then, because of (3.4)

$$G_k(x) = \prod_{\alpha \in R^+} (x^{-\alpha/2} - x^{\alpha/2}) \prod_{\alpha \in R} (qx^\alpha)_{k-1}. \tag{3.6}$$

We claim that $G_k(x)$ is antisymmetric. Indeed, the second product is symmetric because any element $w \in W$ sends R to itself [C, p. 13, line 4] and the first product is antisymmetric for the same reason, only now we get a minus sign whenever a positive root α is sent to a negative root (i.e., whenever $w(\alpha) \in R^-$). Thus the effect of applying $w \in W$ on the first product of (3.6) is to multiply it by $(-1)^{n(w)}$ where $n(w) = |w(R^+) \cap R^-|$ and this is equal to the sign of w [C, p. 18]. It thus follows that $G_k(x)$ itself is antisymmetric.

From now on, we will forget all about F_k (and certainly about F'_k) and work solely with G_k, noting that the quantity of interest, H_k, is given in terms of G_k by

$$H_k = \text{C.T.}\,(x^\delta G_k) C_{x^{-\delta}} G_k. \tag{3.7}$$

Our goal, to be pursued in the next two sections, is to find H_k. But we must show now, following Stembridge [Ste3], that H_k is indeed related to H'_k as promised by (3.3).

Indeed, since $(y)_k = (y)_{k-1}(1-q^k y)$, we have (by (3.1) and (3.2))

$$\begin{aligned} H'_k &= \text{C.T.} \prod_{\alpha\in R^+} (1-q^k x^{-\alpha}) F_k = \text{C.T.} \left[\prod_{\alpha \in R^+} (1-q^k x^{-\alpha}) \right] x^\delta G_k \\ &= \text{C.T.} \prod_{\alpha\in R^+} [(1-q^k x^{-\alpha}) x^{\alpha/2}] G_k = \text{C.T.} \prod_{\alpha \in R^+} (x^{\alpha/2} - q^k x^{-\alpha/2}) G_k. \end{aligned} \tag{3.8}$$

When the product on the extreme right is expanded we get $2^{|R^+|}$ terms, since each term in the product corresponds to a pair of opposite roots, and each term in the resulting huge sum corresponds to choosing, for every α in R^+, whether to take it or its negative. This prompts us to define a *choice set* Ω, as a subset of R such that for each $\alpha \in R^+$ either $\alpha \in \Omega$ or $-\alpha \in \Omega$.

We can now write the right-hand side of (3.8) as (set $t = q^k$),

$$\text{C.T.} \sum_{\Omega\ \text{choice set}} (-t)^{|\Omega \cap R^-|} (x^{\text{sum}(\Omega)/2} G_k). \tag{3.9}$$

Here sum (Ω) denotes the (vector) sum of all the elements in Ω.

Now let us call a choice set a *bad guy*, if sum (Ω) lies on a reflecting hyperplane, i.e., there exists a root β such that $(\text{sum}(\Omega), \beta) = 0$. Otherwise let us call it a *good guy*. The sum in (3.9) can, of course, be written as

$$\text{C.T.} \sum_{\Omega\ \text{good guy}} (-t)^{|\Omega\cap R^-|} (x^{\text{sum}(\Omega)/2} G_k) + \text{C.T.} \sum_{\Omega\ \text{bad guy}} (-t)^{|\Omega \cap R^-|} (x^{\text{sum}(\Omega)/2} G_k). \tag{3.10}$$

The proof of (3.3) will continue *right after this.*

CRUCIAL LEMMA. *Let $G(x)$ be antisymmetric with respect to the Weyl group W and let γ be any vector of integers.*

(i) $\text{C.T.}\,(x^{w(\gamma)} G) = \text{sgn}\,(w)\, \text{C.T.}\,(x^\gamma G)$, *for each element w in the Weyl group W.*

(ii) *If γ lies on a reflecting hyperplane, i.e., there exists an $\alpha \in R$ such that $(\gamma, \alpha) = 0$, then* $\text{C.T.}\,(x^\gamma G) = 0$.

Proof of the Crucial Lemma.

Proof of (i).

$$\begin{aligned} \text{C.T.}\,(x^{w(\gamma)} G) &= \text{C.T.}\, w(x^\gamma w^{-1}(G)) \\ &= \text{C.T.}\, x^\gamma w^{-1}(G) = \text{C.T.}\, x^\gamma\, \text{sgn}\,(w^{-1}) G \\ &= \text{sgn}\,(w)\, \text{C.T.}\, x^\gamma G. \end{aligned}$$

In this chain of equalities we have used, in that order: (a) the definition of the action of w on a Laurent polynomial; (b) the fact that applying w on a Laurent polynomial never changes the consant term (because w is, among other things, a linear transformation, so $w(0) = 0$ and $w(x^0) = x^{w(0)} = x^0$); (c) the antisymmetry of G; (d) $\text{sgn}\,(w^{-1}) = \text{sgn}\,(w)$, and you can always take a constant out of C.T.

Proof of (ii). Let w_α be the Weyl reflection corresponding to the root α [C, p. 12]; then $w_\alpha(\gamma) = \gamma$ (since γ lies on the mirror that is perpendicular to α) and since $\text{sgn}\,(w_\alpha) = -1$, we have, by part (i),

$$\text{C.T.}\,(x^\gamma G) = \text{C.T.}\,(x^{w_\alpha(\gamma)} G) = (\text{sgn}\,(w_\alpha))\, \text{C.T.}\,(x^\gamma G) = -\text{C.T.}\,(x^\gamma G).$$

Thus $\text{C.T.}\,(x^\gamma G)$ is equal to its negative and must be zero. □

We now return to the proof of (3.3).

Because of part (ii) of the Crucial Lemma and the definition of a bad guy, the second sum in (3.10) vanishes. Now from Lemma 10.1.6. and its proof of [C, p. 147], or from Lemma 2.13 of [Ma2], it follows that if Ω is a good guy then there exists a w in the Weyl group W such that $\mathrm{sum}\,(\Omega)/2 = w(\delta)$ and $\Omega = w(R^+)$. So a good choice set Ω uniquely determines $w \in W$ and vice versa. It is thus possible to write (3.9) as (note that $|w(R^+) \cap R^-| = n(w)$)

$$H'_k = \sum_{w \in W} (-1)^{n(w)} t^{n(w)}\, \mathrm{C.T.}\,(x^{w(\delta)} G_k).$$

But because of part (i) of the Crucial Lemma,

$$\mathrm{C.T.}\,(x^{w(\delta)} G_k) = \mathrm{sgn}\,(w)\, \mathrm{C.T.}\,(x^{\delta} G_k)$$

and since $\mathrm{sgn}\,(w) = (-1)^{n(w)}$, we have that H'_k is equal to

$$H'_k = \left(\sum_{w \in W} t^{n(w)} \right) H_k,$$

and (3.3) follows because of the following beautiful identity due to Bott, Solomon, and Macdonald [Ma2] (see [C, p. 135 ff, p. 155])

$$\sum_{w \in W} t^{n(w)} = \prod_{i=1}^{l} \frac{1 - t^{d_i}}{1 - t}. \tag{3.11}$$

We should remark, though, that if one is only interested in one root system at a time (as we are in the present method), then we really do not need (3.11), since the left-hand side is just a specific polynomial that can be explicitly computed and, if desired, factorized.

4. Induction. This section constitutes my own twist on the Stembridge approach. Stembridge's [Ste3] inductive scheme, for A_n, was to creep along the coefficients of G_k (keeping k fixed) until one gets to a high enough coefficient whose value is equal to the H_k for A_{n-1}. So his induction was with respect to n, and his k stayed fixed. Our induction is with respect to k and the root system stays fixed.

Using $(y)_{k+1} = (1 - q^k y)(y)_k$, $(qy)_k = (1 - q^k y)(qy)_{k-1}$, (3.2) and (3.5), we have

$$H_{k+1} = \mathrm{C.T.}\,(x^{\delta} G_{k+1}) = \mathrm{C.T.}\left(x^{\delta} \prod_{\alpha \in R} (1 - q^k x^{\alpha}) G_k \right). \tag{4.1}$$

Now put $t = q^k$ and expand the product

$$x^{\delta} \prod_{\alpha \in R} (1 - t x^{\alpha}) = \sum_{\rho' \in S'} a_{\rho'}(t) x^{\rho'} \tag{4.2}$$

where S' is a certain *finite* set of vectors in the lattice generated by the roots and $a_{\rho'}(t)$ are polynomials in t. Now, each $\rho' \in S'$ is either on a reflecting hyperplane (a bad guy) or [C, Prop. 2.3.4, p. 22] there is a $w \in W$ and ρ in the fundamental chamber such that $\rho' = w(\rho)$. Thus defining S to be the set of all the W images of S' that lie in the fundamental chamber, the right-hand side of (4.2) can be written as

$$\sum_{\rho'\,bad} a_{\rho'}(t) x^{\rho'} + \sum_{\rho \in S} \sum_{w \in W} a_{\rho,w}(t) x^{w(\rho)} \tag{4.3}$$

where $a_{\rho,w}(t)$ are certain (easily computable) polynomials in t (some of which may be zero).

Substituting this into the right-hand side of (4.1) we get that the contribution from the first sum in (4.3) is zero (Crucial Lemma (ii)), and it follows from part (i) of the Crucial Lemma that

$$H_{k+1}=\sum_{\rho\in S}\sum_{w\in W} a_{\rho,w}(t)\,\text{C.T.}\,(x^{w(\rho)}G_k)=\sum_{\rho\in S}\left(\sum_{w\in W} a_{\rho,w}(t)\,\text{sgn}\,(w)\right)\text{C.T.}\,(x^{\rho}G_k). \tag{4.4}$$

Now for each $\rho\in S$, let

$$A_\rho(t)=\sum_{w\in W} a_{\rho,w}(t)\,\text{sgn}\,(w). \tag{4.5}$$

$A_\rho(t)$ is a certain explicitly computable polynomial in t. Going back to (4.4) we have

$$H_{k+1}=\sum_{\rho\in S} A_\rho(t)\,\text{C.T.}\,(x^{\rho}G_k). \tag{4.6}$$

One of the summands here is $\rho=\delta$, so we have expressed H_{k+1} in terms of C.T. $(x^{\delta}G_k)=H_k$ and a certain *finite* number of "neighboring coefficients." We have thus encountered the notorious "problem of uninvited guests" that crops up so often when trying to prove something by induction. One way out of this, the polite way, is to put up with these undesirable terms and conjecture that they too, have a certain explicit form, and then redo (4.6) to account for these as well (and cross our fingers that they will not bring in more undesirable terms). I do not see how to do it (at least not yet). The other way is the rude way. Get rid of these undesirable terms by expressing all of them in terms of the only term that we really care about: the one and only H_k.

5. Equations. This section will describe Stembridge's variation on an old trick in q-series, adapted to our needs. This trick converts a q-product in one variable $f(x)$ into a sum by computing $f(qx)/f(x)$. If this turns out to be a rational function, then cross-multiplying yields a functional equation relating $f(x)$ and $f(qx)$. By expanding $f(x)$ in a power series, this translates into a linear recurrence in the coefficients, that sometimes can be solved explicitly. However, attempting to use this method for multivariate products always produces a mess, unless we have antisymmetry on our side, and even then one has to be very careful.

So let us go to business. Using the definitions (3.2) and (3.5), we have

$$G_k(x)=x^{-\delta}\prod_{\alpha\in R^+}(x^{\alpha})_k(qx^{-\alpha})_{k-1}. \tag{5.1}$$

Recall that $x=(x_1,\cdots,x_l)$, $\alpha=(\alpha_1,\cdots,\alpha_l)$ and $x^{\alpha}=x_1^{\alpha_1}\cdots x_l^{\alpha_l}$. Define

$$f_\alpha(x)=(x^{\alpha})_k(qx^{-\alpha})_{k-1};$$

then if $\alpha_1=0, f_\alpha(x_1\leftarrow qx_1)=f_\alpha(x)$, and in general (we assume, without loss of generality (see Introduction) that α has integer coordinates)

$$\frac{f_\alpha(x_1\leftarrow qx_1)}{f_\alpha(x)}=\frac{(q^{\alpha_1}x^{\alpha})_k(q^{1-\alpha_1}x^{-\alpha})_{k-1}}{(x^{\alpha})_k(qx^{-\alpha})_{k-1}}. \tag{5.2}$$

Now by making all the $(\,)_k$ explicit and using telescoping, we easily obtain

$$\frac{f_\alpha(x_1\leftarrow qx_1)}{f_\alpha(x_1)}=\frac{p_\alpha(x)}{q_\alpha(x)} \tag{5.3}$$

where, if $\alpha_1>0$,

$$\begin{aligned} p_\alpha(x)&=(1-q^{k}x^{\alpha})\cdots(1-q^{k+\alpha_1-1}x^{\alpha}),\\ q_\alpha&=(q^{k-1}-x^{\alpha})\cdots(q^{k-1}-q^{\alpha_1-1}x^{\alpha}),\end{aligned} \tag{5.4a}$$

and if $\alpha_1 < 0$,

$$(5.4b) \qquad \begin{aligned} p_\alpha(x) &= (q^{k-1} - q^{-1}x^\alpha) \cdots (q^{k-1} - q^{\alpha_1}x^\alpha), \\ q_\alpha(x) &= (1 - q^{k-1}x^\alpha) \cdots (1 - q^{k+\alpha_1}x^\alpha). \end{aligned}$$

Since for all root-systems [C, pp. 47–49] $-2 \leqq \alpha_1 \leqq 2$, p_α, q_α, are at worst quadratic in x^α.

It follows from (5.3) and the definition (5.1) that

$$(5.5) \qquad \frac{G_k(x_1 \leftarrow qx_1)}{G_k(x_1)} = q^{-\delta_1} \prod_{\alpha \in R^+} \frac{p_\alpha(x)}{q_\alpha(x)} = \frac{P(x, q^k, q)}{Q(x, q^k, q)},$$

say, where P and Q are certain, explicitly computable, polynomials in $x = (x_1, \cdots, x_l)$, q^k and q.

Now, by cross-multiplying (5.5), we get the functional equation

$$(5.6) \qquad Q(x)G_k(x_1 \leftarrow qx_1) = P(x)G_k(x).$$

Out of this functional equation we can get many linear equations relating various coefficients of G_k. For any vector β in the lattice generated by R, we will get a linear equation E_β, involving coefficients C.T. $(x^\gamma G_k)$ for γ in a certain set of vector exponents $Ex(\beta)$, that is contained in the fundamental chamber.

The way to do this is to first multiply both sides of (5.6) by x^β and then apply the functional C.T.

$$(5.7) \qquad \text{C.T.}\,[x^\beta Q(x)G_k(x_1 \leftarrow qx_1)] = \text{C.T.}\,[x^\beta P(x)G_k(x)].$$

We now plug into (5.7) the expanded form of P and Q (remember that P and Q are certain explicit polynomials that we have to compute in order to perform the algorithm). Then we use the linearity of C.T. and get on the right-hand side a linear combination of creatures of the form C.T. $[x^\gamma G_k]$. On the left-hand side we get a linear combination of entities of the form C.T. $[x^\gamma G_k(x_1 \leftarrow qx_1)]$. These should be converted to the previous form using the obvious relation

$$(5.8) \qquad \text{C.T.}\,[x^\gamma G_k(x_1 \leftarrow qx_1)] = q^{-\gamma_1}\,\text{C.T.}\,[x^\gamma G_k].$$

We now use the Crucial Lemma, discarding all the "bad" γ, i.e., those that are orthogonal to a root, and for any good γ' that is not in the fundamental chamber we find the unique $w \in W$ and γ in the fundamental chamber such that $\gamma' = w(\gamma)$ and rewrite C.T. $[x^{\gamma'} G_k]$ as sgn (w) C.T. $[x^\gamma G_k(x)]$. Then we collect all the terms and bring them to the left-hand side and get a certain linear equation

$$(5.9) \qquad E_\beta\colon \quad \sum a_\gamma(q^k, q)\,\text{C.T.}\,(x^\gamma G_k) = 0$$

where the sum is over a *finite* set $Ex(\beta)$ of exponents γ that lie in the fundamental chamber.

By a judicious choice of β we would hopefully obtain equations that only involve those $\rho \in S$ that feature in (4.6). Hopefully there would be $|S| - 1$ such independent equations. (Of course it would also be all right if we could say the same thing about some set that contains S.) By a proper choice of β it is always possible to get an equation that involves C.T. $(x^\delta G_k) = H_k$.

Solving this system of $|S| - 1$ homogeneous equations, at least one of which involves H_k, we should be able to express all the unknowns as H_k times some rational function in q^k and q. This is so since the coefficients in the system are polynomials in q^k and q. We have thus found explicit expressions for all the terms that feature in (4.6) in terms of H_k, and plugging them in we will get H_{k+1}/H_k, a certain rational function in q^k and q. Calling the conjectured value of H_k by the name of R_k (R_k is the right-hand side of (qM) divided by (3.11)), we can then compare H_{k+1}/H_k with R_{k+1}/R_k. Since obviously $H_1 = R_1$, the fate of (qM) will be determined by whether or not H_{k+1}/H_k is equal to R_{k+1}/R_k.

6. Implementation. This can be, and has been, implemented on a computer. The input is the root system R, and it is necessary to know the Weyl group W (this is given in the *planches* of [Bo]). It is very easy to write a routine to check whether a given vector is a *bad guy* (just do-loop the inner product along R^+). Then you need to write a Weyl-sorting routine that given any good vector in the root-lattice finds its image in the fundamental chamber and the sign of the element w in W that sends it there. Of course you also need a polynomial multiplication routine (which you can easily jot down yourself, no need for MACSYMA). This is enough to produce (4.6) and the $P(x)$ and $Q(x)$ of (5.5).

Now comes the creative part, experimenting with various β's that will give an equation E_β that involves the relevant coefficients that feature in (4.6). For those root systems for which $-1 \leqq \alpha_1 \leqq 1$ (most of them) the choice $\beta = -\delta$ will produce a tautology: $0=0$, because the only survivor, after applying part (ii) of the Crucial Lemma, is C.T. $[x^\delta G_k] = H_k$. It is thus likely that for β near $-\delta$ we will get relatively few terms.

Once you have $|S|-1$ independent equations you solve them and plug the solutions into (4.6). You will never have to see (or print out) the solutions of the system (5.9), because it can all be done internally (in MACSYMA this amounts to finishing your lines with dollar signs rather than with semicolons). You will not even have to see or print out the resulting rational function H_{k+1}/H_k obtained by plugging in the solutions of the system (5.9) into (4.6).

All you have to do is enter the rational function R_{k+1}/R_k (you can even write a routine for that) and ask the computer to output the difference between these two rational functions. If you get ZERO then you have proved (qM) for your particular root system. If you get something else then you have *disproved* (qM). Either that or (more likely), you have made an error somewhere.

7. A_2. The new method will now be illustrated on the simplest nontrivial case, the root system A_2. Of course this case is already well known, even classical (it is equivalent to Jackson's q-Dixon identity [An1]), and the proof that we present here is perhaps the longest and ugliest ever. But in order to learn how to use machine guns to kill elephants one should first practise on flies. Another reason for doing the A_2 case is that its results will be needed in § 9, when we do $G_2^\vee$, and this will make the paper self-contained. The present example is simple enough that it can be done by hand, and the reader is encouraged to check all the steps and to supply all the details.

Equation (qM) says, in its equivalent formulation derived in § 3, that if

$$F_k = \left(\frac{x_1}{x_2}\right)_k \left(\frac{x_1}{x_3}\right)_k \left(\frac{x_2}{x_3}\right)_k \left(q\frac{x_2}{x_1}\right)_{k-1} \left(q\frac{x_3}{x_1}\right)_{k-1} \left(q\frac{x_3}{x_2}\right)_{k-1},$$

$$H_k = \text{C.T. } F_k$$

and

$$R_k = \frac{(q)_{3k-1}}{(q)_{k-1}^2 (q)_k (1-q^{2k})},$$

then $H_k = R_k$.

To get R_k from $R'_k=(q)_{3k}/(q)_k^3$ we used (3.3) with the fundamental invariants 2, 3 of A_2. A list of the fundamental invariants for all finite irreducible root systems can be found for example in the excellent appendices of [Bo], as well as in [C, p. 155].

Now a routine calculation shows that ($t=q^k$),

$$\frac{R_{k+1}}{R_k}=\frac{(1+t+t^2)(1-qt^3)(1-q^2t^3)(1+t)}{(1-qt)(1-q^2t^2)}. \tag{7.1}$$

Now it is easily checked that $R_1=1$ and $H_1=1$, so all we have to do is verify that H_{k+1}/H_k is equal to R_{k+1}/R_k. So let us compute H_{k+1}/H_k.

For A_2 we have (e.g., [Bo, p. 250] or [C, p. 46])

$$A_2^+=\{(1,-1,0);\,(1,0,-1);\,(0,1,-1)\}.$$

$\delta=(1,0,-1)$, and the Weyl group W is S_3, the symmetric group on three elements that acts by permuting the coordinates of $(\gamma_1,\gamma_2,\gamma_3)$ for γ in the root lattice. The bad guys are those vectors that have two of their coordinates equal.

Now we do (4.2), namely we expand

$$\frac{x_1}{x_3}\left(1-t\frac{x_1}{x_2}\right)\left(1-t\frac{x_2}{x_1}\right)\left(1-t\frac{x_1}{x_3}\right)\left(1-t\frac{x_3}{x_1}\right)\left(1-t\frac{x_2}{x_3}\right)\left(1-t\frac{x_3}{x_2}\right).$$

Discarding the bad guys, grouping the good guys into orbits under S_3, as in (4.3), plugging into (4.1) and using the Crucial Lemma yields, like in (4.4)-(4.6) (set $A(\rho)=\text{C.T.}\,[x^\rho G_k]$),

$$\begin{aligned}H_{k+1}=&(1+2t+3t^2+3t^3+3t^4+2t^5+t^6)A(1,0,-1)\\&-(t+t^2+2t^3+t^4+t^5)A(2,0,-2)\\&+(t^2+t^3+t^4)A(2,1,-3)+(t^2+t^3+t^4)A(3,-1,-2)-t^3A(3,0,-3).\end{aligned} \tag{7.2}$$

Thus, $S=\{(1,0,-1),\,(2,0,-2),\,(2,1,-3),\,(3,-1,-2),\,(3,0,-3)\}$, and we need to find four independent equations relating $\{A(\rho);\,\rho\in S\}$.

Now (5.3) becomes

$$\frac{f_{1-10}(qx_1,x_2,x_3)}{f_{1-10}(x_1,x_2,x_3)}=\frac{\left(1-q^k\frac{x_1}{x_2}\right)}{\left(q^{k-1}-\frac{x_1}{x_2}\right)},\qquad \frac{f_{10-1}(qx_1,x_2,x_3)}{f_{10-1}(x_1,x_2,x_3)}=\frac{\left(1-q^k\frac{x_1}{x_3}\right)}{\left(q^{k-1}-\frac{x_1}{x_3}\right)},$$

and (5.5) becomes ($\delta_1=1$),

$$\frac{G_k(qx_1,x_2,x_3)}{G_k(x_1,x_2,x_3)}=q^{-1}\frac{\left(1-q^k\frac{x_1}{x_2}\right)\left(1-q^k\frac{x_1}{x_3}\right)}{\left(q^{k-1}-\frac{x_1}{x_2}\right)\left(q^{k-1}-\frac{x_1}{x_3}\right)},$$

and (5.6) becomes

$$q\left(q^{k-1}-\frac{x_1}{x_2}\right)\left(q^{k-1}-\frac{x_1}{x_3}\right)G_k(qx_1,x_2,x_3)=\left(1-q^k\frac{x_1}{x_2}\right)\left(1-q^k\frac{x_1}{x_3}\right)G_k(x_1,x_2,x_3)$$

and multiplying out yields

$$\begin{aligned}&\left(q^{2k-1}-q^k\frac{x_1}{x_2}-q^k\frac{x_1}{x_3}+q\frac{x_1^2}{x_2x_3}\right)G_k(qx_1,x_2,x_3)\\&=\left(1-q^k\frac{x_1}{x_2}-q^k\frac{x_1}{x_3}+q^{2k}\frac{x_1^2}{x_2x_3}\right)G_k(x_1,x_2,x_3).\end{aligned} \tag{7.3}$$

Experimenting with various β yields that $(0,1,-1)$, $(1,0,-1)$, $(0,2,-2)$, and $(1,1,-2)$ produce the desired equations (of course there are many other choices of β that will do). For each of these β, multiplying both sides of (7.3) by x^β, using (5.8) and the Crucial Lemma yields the equations:

$$E_{(0,1,-1)}\colon (1-q^{2k-1}-q^{k-1}+q^k)A(1,0,-1)+(q^{-1}-q^{2k})A(2,0,-2)=0,$$

$$E_{(1,0,-1)}\colon (1-q^{2k-2})A(1,0,-1)+(q^{k-2}-q^k)A(2,0,-2)+(q^{2k}-q^{-2})A(3,-1,-2)=0,$$

$$E_{(0,2,-2)}\colon (1-q^{2k-1})A(2,0,-2)+(q^{-1}+q^{k-1}-q^k-q^{2k})A(2,1,-3)=0,$$

$$E_{(1,1,-2)}\colon (q^{k-2}-q^k)A(2,0,-2)+(q^{k-2}-q^k)A(2,1,-3)+(q^{2k}-q^{-2})A(3,0,-3)=0.$$

Solving this system we get $(t=q^k)$ (recall that $A(1,0,-1)=H_k$),

$$A(2,0,-2)=\frac{(t-q)(1-t^2)}{(1-t)(1-qt^2)}H_k,$$

(7.4)
$$A(3,-1,-2)=A(2,1,-3)=\frac{(q-t)(q-t^2)}{(1-qt)(1-qt^2)}H_k,$$

$$A(3,0,-3)=\frac{-t(1-q^2)(q-t)(1-q)(1-t^3)}{(1-qt)(1-qt^2)(1-q^2t^2)(1-t)}H_k.$$

This much was done by hand. Now using MACSYMA we can plug it all into (7.2) and get H_{k+1}/H_k. Then we subtract it from R_{k+1}/R_k given in (7.1). The answer is indeed zero and we have just proved (qM) for A_2.

Now that we know that H_k is indeed equal to what it is supposed to be, namely to R_k, we can plug that expression into (7.4) and get as a lagnappe explicit expressions for $A(2,0,-2)=A_k(2,0,-2)=\text{C.T.}\,[x_1^2x_3^{-2}G_k]=\text{C.T.}\,[x_1x_3^{-1}F_k]$, etc. This will be needed in § 9.

8. Modifications. Our method can be easily adapted to the more refined Macdonald–Morris conjectures (qM-M1) and (qM-M2). In fact, because of the added parameter it is even computationally faster. We will only treat (qM-M2), since (qM-M1) is just a special case of (qM-M2) ((qM-M1) corresponds to the $S(R)$ cases for which it is well known [Ma1], [Mo] that $u_\alpha\equiv 1$). It is also well known (for example from the classification theorem for finite root systems [Bo], [C] that all the irreducible reduced finite root systems have either just one root length (A_n, D_n, E_6, E_7, and E_8) or two root lengths (B_n, C_n, G_2, and F_4). The only nonreduced irreducible finite root systems, BC_n, have three different root lengths. Since (qM-M2) reduces to (qM) for all the single-length root systems, we will assume that the root systems have two root lengths, short and long, and leave it to the reader to do the appropriate obvious modifications for BC_n.

So let us rewrite (qM-M2) for two-lengths root systems. Denoting k_{short} by a, k_{long} by b, u_{short} by u_s, and u_{long} by u_l, we have

(qM-M2′)
$$\text{C.T.}\left\{\prod_{\alpha\in R^+_{\text{short}}}(x^\alpha;q^{u_s})_a(q^{u_s}x^{-\alpha};q^{u_s})_a\prod_{\alpha\in R^+_{\text{long}}}(x^\alpha;q^{u_l})_b(q^{u_l}x^{-\alpha};q^{u_l})_b\right\}$$
$$=\text{a certain explicit product.}$$

The right-hand side can be looked up in [Ma3] or [Mo, pp. 25–26]. Its exact form is irrelevant for the purposes of the present method whose modest aim is to prove (qM-M2) for one root-system *at a time*, and as such does not care to look at the general pattern. Besides, the method should be able to compute the constant term in question *from scratch* and it is dishonest to "peek at the answer." In any case, for any specific root-system, it is possible to look up the explicit conjectured right-hand side from [Mo].

So let us call the polynomial inside the braces of (qM-M2′) $F'_{a,b}(x)$. We are interested in evaluating

$$H'_{a,b} = \text{C.T. } F'_{a,b}(x). \tag{8.1}$$

In analogy with § 3, we will consider instead

$$F_{a,b}(x) = \prod_{\alpha\in R^+_{\text{short}}} (x^\alpha; q^{u_s})_a (q^{u_s} x^{-\alpha}; q^{u_s})_{a-1} \prod_{\alpha\in R^+_{\text{long}}} (x^\alpha; q^{u_l})_b (q^{u_l} x^{-\alpha}; q^{u_l})_{b-1},$$

$$H_{a,b} = \text{C.T. } F_{a,b}(x).$$

Since the Weyl group W acts separately on the long roots and the short roots (as is obvious from the fact that the elements of W are isometries), the calculation of (3.3) can be carried verbatim to show that

$$G_{a,b}(x) =: x^{-\delta} F_{a,b}(x) \tag{8.2}$$

is antisymmetric.

For w in the Weyl group W let $n_s(w) = |w(R^+_{\text{short}}) \cap R^-|$ and $n_l(w) = |w(R^+_{\text{long}}) \cap R^-|$ (so $n(w) = n_s(w) + n_l(w)$). Define

$$W(t, s) = \sum_{w\in W} t^{n_s(w)} s^{n_l(w)}, \tag{8.3}$$

which for a fixed root system (and therefore a fixed Weyl group) is a specific polynomial. Macdonald [Ma2] has a wonderful formula for $W(t, s)$ as a product that is indexed over the positive roots (for $t = s$ it reduces to (3.11)), but it is not really needed for our present narrow-minded purposes.

A completely analogous argument to that of § 3 (only now we keep track of the short and long roots separately, with their respective parameters t and s) yields

$$H'_{a,b} = H_{a,b} W(q^{au_s}, q^{bu_l}). \tag{8.4}$$

We now want to evaluate

$$H_{a,b} = \text{C.T. } [x^\delta G_{a,b}].$$

The difference now is that we have two parameters a and b rather than the single parameter k. The induction step is similar to that described in § 4 only now we induct with respect to either *a or b* (I prefer a). Unlike the previous case where the base case was trivial, now $a = 1$ is no longer trivial but is essentially the (qM) conjecture for the subroot system consisting of the long roots

$$H_{1,b} = \text{C.T. } F_{1,b}(x)$$

where

$$F_{1,b}(x) = \prod_{\alpha\in R^+_{\text{short}}} (1 - x^\alpha) \prod_{\alpha\in R^+_{\text{long}}} (x^\alpha; q^{u_l})_b (q^{u_l} x^{-\alpha}; q^{u_l})_{b-1}. \tag{8.5}$$

Expanding

$$\prod_{\alpha \in R^+_{\text{short}}} (1-x^\alpha),$$

we can express $H_{1,b}$ as a certain linear combination of various coefficients of

$$\hat{F}_b(x) := \prod_{\alpha \in R^+_{\text{long}}} (x^\alpha; q^{u_l})_b (q^{u_l} x^{-\alpha}; q^{u_l})_{b-1}.$$

Thus before we can embark on (qM-M2) for R we must do first (qM) for the subroot system R_{long} and find not only the constant term of $\hat{F}_b(x)$ but also some neighboring coefficients. It can be easily shown that these coefficients are among those "lagnappes" that we got anyway in system (5.9). For example, the long roots of G_2 constitute the root system A_2, and when we do $G_2^\vee$ in the next section we will use the A_2 information obtained in § 7. Similarly, before we can do F_4 we must do D_4, etc.

Having established the base case $a=1$, § 4 passes almost verbatim: (4.1) becomes

$$H_{a+1,b} = \text{C.T.}\,[x^\delta G_{a+1,b}(x)] = \text{C.T.}\left[x^\delta \prod_{\alpha \in R_{\text{short}}} (1-q^{au_s} x^\alpha) G_{a,b}\right] \tag{8.6}$$

and (4.6) becomes $(t = q^{au_s})$

$$H_{a+1,b} = \sum_{\rho \in S} A_\rho(t)\, \text{C.T.}\,[x^\rho G_{a,b}]. \tag{8.7}$$

Now comes the analogue of § 5. We have to be a little careful because $G_{a,b}(x_1 \leftarrow q x_1)/G_{a,b}(x)$ may not be a rational function. Instead we look for vectors of integers $z = (z_1, \cdots, z_n)$ such that

$$\frac{G_k(q^{z_1} x_1, \cdots, q^{z_n} x_n)}{G_k(x_1, \cdots, x_n)} \tag{8.8}$$

is a rational function. This can be achieved if u_s divides (α, z) for every short root α, and u_l divides (α, z) for every long α. Of course we will try to choose z in such a way that the rational function (8.8) is as simple as possible (in the next section $z = (2, 1, 0)$).

In analogy with § 5 we define

$$f_\alpha(x) = (x^\alpha; q^{u_\alpha})_{k_\alpha} (q^{u_\alpha} x^{-\alpha}; q^{u_\alpha})_{k_\alpha - 1}$$

where $k_\alpha = a$, $u_\alpha = u_s$ if α is short and $k_\alpha = b$ and $u_\alpha = u_l$ if α is long.

In analogy with (5.2) we have

$$\frac{f_\alpha(q^{z_1} x_1, \cdots, q^{z_n} x_n)}{f_\alpha(x_1, \cdots, x_n)} = \frac{(q^{(z,\alpha)} x^\alpha; q^{u_\alpha})_{k_\alpha} (q^{u_\alpha - (z,\alpha)} x^{-\alpha}; q^{u_\alpha})_{k_\alpha - 1}}{(x^\alpha; q^{u_\alpha})_{k_\alpha} (q^{u_\alpha} x^{-\alpha}; q^{u_\alpha})_{k_\alpha - 1}}.$$

So if we replace

$$q \leftarrow q^{u_\alpha}, \qquad \alpha_1 \leftarrow (z, \alpha)/u_\alpha, \qquad k \leftarrow k_\alpha$$

then (5.3) and (5.4) are still true. Equation (5.5) now becomes

$$\frac{G_{a,b}(q^{z_1} x_1, \cdots, q^{z_n} x_n)}{G_{a,b}(x_1, \cdots, x_n)} = q^{-(\delta, z)} \prod_{\alpha \in R^+} \frac{p_\alpha(x)}{q_\alpha(x)} = \frac{P}{Q} \tag{8.9}$$

where P and Q are explicitly computable polynomials in x, q, t, and s where $t = q^{au_s}$; $s = q^{bu_l}$.

Equations (5.6) and (5.7) are still true but with $G_k(x_1 \leftarrow qx_1)$ replaced by $G_k(q^{z_1}x_1, \cdots, q^{z_n}x_n)$ and instead of (5.8) we need

$$\text{C.T.}\,[x^\gamma G_{a,b}(q^{z_1}x_1, \cdots, q^{z_n}x_n)] = q^{-(\gamma,z)}\,\text{C.T.}\,[x^\gamma G_{a,b}]. \tag{8.10}$$

This follows from

$$\begin{aligned}&\text{C.T.}\,[x^\gamma G_{a,b}(q^{z_1}x_1, \cdots, q^{z_n}x_n)]\\ &\quad = q^{-(\gamma,z)}\,\text{C.T.}\,[(q^{z_1}x_1)^{\gamma_1}\cdots(q^{z_n}x_n)^{\gamma_n}\cdots G_{a,b}(q^{z_1}x_1, \cdots, q^{z_n}x_n)]\end{aligned}$$

and the fact that C.T. is unaffected by scaling.

Everything is as before; the only difference is that in (5.9) the coefficients a_γ depend on (t, s, q) where $t = q^{au_s}$ and $s = q^{bu_l}$, i.e.,

$$E_\beta\colon\ \sum a_\gamma(t, s, q)\,\text{C.T.}\,[x^\gamma G_{a,b}] = 0. \tag{8.11}$$

Everything else translates smoothly. Solving the system we will express all the coefficients that feature in (8.7) as certain rational functions in (q, t, s) times $\text{C.T.}\,[x^\gamma G_{a,b}] = H_{a,b}$. Substituting the solutions thus obtained into (8.7) will give us the rational function $H_{a+1,b}/H_{a,b}$. Since we already have a formula for $H_{1,b}$ this easily yields a formula for $H_{a,b}$. Alternatively, if we believe that the conjectured value for $H_{a,b}$, let us call it $R_{a,b}$, has a good chance of being correct then all we have to do is look up $R'_{a,b}$ (the conjectured right-hand side of (qM-M2)) in [Mo] and then compute $R_{a,b}$ by dividing $R'_{a,b}$ by $W(q^{au_s}, q^{bu_l})$ of (8.3). We then compute $R_{a+1,b}/R_{a,b}$ (a rational function in (q, t, s)). Assuming that we have already checked that $H_{1,b} = R_{1,b}$, the status of the conjecture (qM-M2) for the particular root-system in question is determined by whether or not $H_{a+1,b}/H_{a,b} - R_{a+1,b}/R_{a,b}$ is zero or not.

9. G_2^ν.

THEOREM (G_2^ν case of (qM-M2)). *The constant term of*

$$\begin{aligned}F'_{a,b}(x,y,z) := {}&\left(\frac{x}{y}; q\right)_a \left(\frac{z}{y}; q\right)_a \left(\frac{z}{x}; q\right)_a \left(\frac{z^2}{xy}; q^3\right)_b \left(\frac{xz}{y^2}; q^3\right)_b \left(\frac{yz}{x^2}; q^3\right)_b\\ &\cdot \left(q\frac{y}{x}; q\right)_a \left(q\frac{y}{z}; q\right)_a \left(q\frac{x}{z}; q\right)_a \left(q^3\frac{xy}{z^2}; q^3\right)_b \left(q^3\frac{y^2}{xz}; q^3\right)_b \left(q^3\frac{x^2}{yz}; q^3\right)_b\end{aligned}$$

is equal to

$$R'_{a,b} := \frac{(q;q)_{3a+3b}(q;q)_{3b}(q;q)_{2a}(q^3;q^3)_{a+3b}(q^3;q^3)_{2b}(q^3;q^3)_a}{(q;q)_{2a+3b}(q;q)_{a+3b}(q;q)_a^2(q^3;q^3)_{a+2b}(q^3;q^3)_{a+b}(q^3;q^3)_b^2}.$$

In this explict form the conjecture appears in Morris' thesis [Mo, p. 139]. It is alluded to in [As3, § 5, fifth sentence] and is mentioned explicitly in [As4].

From [Bo, pp. 274-275] or [Mo] or [C], $G_2^+ = \{(1,-1,0), (0,-1,1), (-1,0,1), (-1,-1,2), (1,-2,1), (-2,1,1)\}$; $\delta = (-1,-2,3)$, and the Weyl group is the dihedral group of order 12, that is the direct product of S_3 with $\{I, -I\}$, where I denotes the identity mapping and $-I(\alpha,\beta,\gamma) = (-\alpha,-\beta,-\gamma)$. (It is a very instructive exericise for you to obtain the Weyl group yourself.) The bad guys are the vectors of integers $(\alpha_1, \alpha_2, \alpha_3)$ in which two coordinates are equal (those that are orthogonal to one of the short roots) and those vectors of integers in which one component is zero (those orthogonal to one of the long roots; recall that for all vectors in the root-lattice the sum of the components is zero).

A direct calculation shows that $W(t, s)$ of (8.3) is given by

$$W(t,s) = 1 + t + s + 2ts + ts^2 + t^2s + 2t^2s^2 + t^3s^2 + t^2s^3 + t^3s^3. \tag{9.1}$$

(For example -231 sends $(1,-1,0)$ to $(1,0,-1)$, a negative root; $(0,-1,1)$ goes to $(1,-1,0)$, a positive root; $(-1,0,1)$ goes to $(0,-1,1)$ a positive root; so $n_s(-231)=|(-231)(R^+)\cap R^-|=|\{(1,0,-1)\}|=1$. Similarly, $n_l(-231)=1$ and so -231 gives a contribution of ts to the sum of (8.3).)

$W(t,s)$ factorizes nicely, namely

$$W(t,s)=(1+t)(1+s)(1+ts+t^2s^2) \tag{9.2}$$

(compare [Ma2, p. 168]).

So with the notation of § 8 it follows from (8.4) that ($u_s=1$, $u_l=3$, $t=q^a$, $s=q^{3b}$)

$$H_{a,b}=\frac{H'_{a,b}}{(1+q^a)(1+q^{3b})(1+q^{a+3b}+q^{2a+6b})}$$

and the theorem will be proved if we can show that $H_{a,b}=R_{a,b}$ where $R_{a,b}=R'_{a,b}/W(q^a,q^{3b})$.

A simple calculation gives that

$$R_{a,b}=\frac{(q;q)_{3a+3b}(q;q)_{3b}(q;q)_{2a-1}(q^3;q^3)_{a+3b-1}(q^3;q^3)_{2b-1}(q^3;q^3)_a}{(q;q)_{2a+3b}(q;q)_{a+3b-1}(q;q)_a(q;q)_{a-1}\cdot(q^3;q^3)_{a+2b}(q^3;q^3)_{a+b}(q^3;q^3)_b(q^3;q^3)_{b-1}}. \tag{9.3}$$

A routine calculation gives ($t=q^a$, $s=q^{3b}$)

$$R_{a+1,b}/R_{a,b}=\frac{(1-qt^3s)(1-q^2t^3s)(1-t^2)(1-qt^2)(1-t^3s^3)(1-q^3t^3)}{(1-qt^2s)(1-q^2t^2s)(1-ts)(1-qt)(1-t)(1-q^3t^3s^2)}. \tag{9.4}$$

So let

$$\begin{aligned} F_{a,b}(x)=&\left(\frac{x}{y};q\right)_a\left(\frac{z}{y};q\right)_a\left(\frac{z}{x};q\right)_a\left(\frac{z^2}{xy};q^3\right)_b\left(\frac{xz}{y^2};q^3\right)_b\left(\frac{yz}{x^2};q^3\right)_b\\ &\cdot\left(q\frac{y}{x};q\right)_{a-1}\left(q\frac{y}{z};q\right)_{a-1}\left(q\frac{x}{z};q\right)_{a-1}\\ &\cdot\left(q^3\frac{xy}{z^2};q^3\right)_{b-1}\left(q^3\frac{y^2}{xz}q^3\right)_{b-1}\left(q^3\frac{x^2}{yz};q^3\right)_{b-1}, \end{aligned} \tag{9.5}$$

$$H_{a,b}=\text{C.T. }F_{a,b}.$$

We must show that $H_{a,b}=R_{a,b}$. This will be done by induction on a. First we must prove the base case $H_{1,b}=R_{1,b}$.

Proof of the base case $a=1$. Substituting $a=1$ in $R_{a,b}$ given in (9.3) and setting $Q=q^3$ gives

$$R_{1,b}=\frac{(Q)_{3b}}{(Q)_b^3}\frac{(1-Q^b)(1-Q)}{(1-Q^{2b})(1-Q^{2b+1})}. \tag{9.6}$$

We will need the A_2 results proved in § 7. For our present purposes it is convenient to rewrite it in the "fundamental roots" form (sometimes used by Morris [Mo]) obtained by setting $u_1=x_1/x_2$ and $u_2=x_2/x_3$. Also let us replace q by Q (so everything is to base Q: $(u_1)_b=(u_1;Q)_b$, etc).

Let $\hat{F}_b$ be defined by

$$\hat{F}_b=(u_1)_b(u_2)_b(u_1u_2)_b(Q/u_1)_{b-1}(Q/u_2)_{b-1}(Q/u_1u_2)_{b-1}. \tag{9.7}$$

Then the results from § 7 that we need here are

$$\text{C.T. } \hat{F}_b = \frac{(Q)_{3b-1}}{(Q)_{b-1}^2 (Q)_b (1-Q^{2b})} \tag{9.8a}$$

and from (7.4) ($A(2,0,-2)$)

$$\text{C.T. } (u_1 u_2 \hat{F}_b) = \frac{(Q^b - Q)(1+Q^b)}{(1-Q^{2b+1})} (\text{C.T. } \hat{F}_b). \tag{9.8b}$$

Now we want $H_{1,b} = \text{C.T. } F_{1,b}$, where (plug in $a=1$ in (9.5)),

$$F_{1,b} = (1-x/y)(1-z/y)(1-z/x)$$
$$\cdot \left(\frac{z^2}{xy}; q^3\right)_b \left(\frac{xz}{y^2}; q^3\right)_b \left(\frac{yz}{x^2}; q^3\right)_b \left(q^3 \frac{xy}{z^2}; q^3\right)_{b-1} \left(q^3 \frac{y^2}{xz}; q^3\right)_{b-1} \left(q^3 \frac{x^2}{yz}; q^3\right)_{b-1}.$$

Now let $u_1 = xz/y^2$, $u_2 = yz/x^2$. Then

$$\frac{x}{y} = u_1^{1/3} u_2^{-1/3}, \qquad \frac{z}{y} = u_1^{2/3} u_2^{1/3}, \qquad \frac{z}{x} = u_1^{1/3} u_2^{2/3},$$

and then if we take $Q = q^3$,

$$F_{1,b} = (1-u_1^{1/3}u_2^{-1/3})(1-u_1^{2/3}u_2^{1/3})(1-u_1^{1/3}u_2^{2/3})\hat{F}_b.$$

So

$$H_{1,b} = \text{C.T. } [1 - u_1^{1/3}u_2^{-1/3} - u_1^{1/3}u_2^{2/3} + u_1 + u_1 u_2 - u_1^{4/3} u_2^{2/3}]\hat{F}_b = \text{C.T. } [(1+u_1 u_2)\hat{F}_b].$$

(u_1 corresponds to the vector $(1,-1,0)+\delta = (1,-1,0)+(1,0,-1) = (2,-1,-1)$, a bad guy (for A_2), and all other terms are even worse: they are fractional. Their contribution is of course zero since $\hat{F}_b$ does not have any terms with fractional exponents, being a Laurent *polynomial.*)

Using (9.8a) and (9.8b) we get

$$H_{1,b} = \left[1 + \frac{(Q^b - Q)(1+Q^b)}{(1-Q^{2b+1})}\right] \frac{(Q)_{3b-1}}{(Q)_{b-1}^2 (Q)_b (1-Q^{2b})}, \tag{9.9}$$

which after a routine calculation turns out to be equal to $R_{1,b}$ in (9.6) (end of proof of the base case $a=1$). □

Proof of the inductive step. Now that we know that $H_{a,b} = R_{a,b}$ for $a=1$ we go next to the inductive step.

For the root system G_2, δ, one-half the sum of the positive roots, is equal to $(-1,-2,3)$, and (8.6) becomes (recall $t = q^a$)

$$\begin{aligned} H_{a+1,b} &= \text{C.T. } [x^{-1}y^{-2}z^3 G_{a+1,b}] \\ &= \text{C.T. } [x^{-1}y^{-2}z^3 (1-tx/y)(1-tz/y)(1-tz/x) \\ &\qquad \cdot (1-ty/x)(1-ty/z)(1-tx/z) G_{a,b}]. \end{aligned} \tag{9.10}$$

We now expand

$$x^{-1}y^{-2}z^3(1-tx/y)(1-tz/y)(1-tz/x)(1-ty/x)(1-ty/z)(1-tx/z),$$

discard all the bad guys and collect all the good guys into orbits under W. We then substitute everything back into (9.10) and use the Crucial Lemma, and then finally we collect terms. (I highly recommend that the reader check this either by hand or by machine, there are only $2^6 = 64$ terms in the expansion.)

We get for (8.7)

$$(9.11)\qquad \begin{aligned} H_{a+1,b} = {} & (1+t+t^2+t^4+t^5+t^6)\,\text{C.T.}\,[x^{-1}y^{-2}z^3G_{a,b}] \\ & -(t+t^3+t^5)\,\text{C.T.}\,[x^{-1}y^{-3}z^4G_{a,b}] \\ & +(t^2+t^3+t^4)\,\text{C.T.}\,[x^{-2}y^{-3}z^5G_{a,b}] - t^3\,\text{C.T.}\,[x^{-1}y^{-4}z^5G_{a,b}]. \end{aligned}$$

In preparation for the MACSYMA *input file* given below let us put

$$\begin{aligned} x0 &= \text{C.T.}\,[x^{-1}y^{-2}z^3G_{a,b}]/H_{a,b} \equiv 1 \quad \text{(by definition)}, \\ x1 &= \text{C.T.}\,[x^{-1}y^{-3}z^4G_{a,b}]/H_{a,b}, \\ x2 &= \text{C.T.}\,[x^{-2}y^{-3}z^5G_{a,b}]/H_{a,b}, \\ x3 &= \text{C.T.}\,[x^{-1}y^{-4}z^5G_{a,b}]/H_{a,b}. \end{aligned}$$

With $p0$, $p1$, $p2$, $p3$ as defined in the *input file* below, (9.11) becomes

$$(9.12)\qquad H_{a+1,b}/H_{a,b} = p0^*x0 + p1^*x1 + p2^*x2 + p3^*x3$$

and we will call this "sum" in the *input file* below.

Finally we need linear equations relating $x0$, $x1$, $x2$, and $x3$. The simplest vector $z = (z_1, z_2, z_3)$ that makes (8.8) a rational function is $(2, 1, 0)$. Now (8.9) becomes

$$\frac{G_{a,b}(q^2x, qy, z)}{G_{a,b}(x, y, z)} = \frac{P}{Q}$$

where P and Q are computed using (5.3) and (5.4) as modified in § 8 before (8.9). Proceeding as described in § 6 (I used a computer but it is possible to do it by hand) we get the following results. ($a00, \cdots, a23$ are given in the MACSYMA *input file* below.) The choice $\beta = (2, 2, -4)$ yields

$$E_{(2,2,-4)}:\quad a00^*x0 + a01^*x1 = 0;$$

$\beta = (0, 3, -3)$ yields

$$E_{(0,3,-3)}:\quad a10^*x0 + a11^*x1 + a12^*x2 = 0;$$

and $\beta = (4, 0, -4)$ yields

$$E_{(4,0,-4)}:\quad a20^*x0 + a21^*x1 + a22^*x2 + a23^*x3 = 0.$$

(A copy of the C program that implements the algorithm of § 5, modified as in § 8, by which I obtained the above equations, is available upon request (either a printout by U.S. mail or by electronic mail; sorry, no disks). However it is highly recommended that the readers write their own programs. It is much easier to write your own code than to try to understand somebody else's computer scratch.)

MACSYMA INPUT FILE

```
a00: t^3*s^2*q - t/q + t*s - t^3*s + t^2*s^2 - t^4*s*q - t^2 + s/q
+ t^2*s*q - t^2*s/q + 1 - t^4*s^2$
a00: 0 - a00$
a01: t^4*s^2*q - q^ - 1$
a10: -t^4*s^2*q + q^2 + t^2*s^2*q + t^2*s^2 - t^4*s*q
```

```
        -t^2*q^3-t^2*q^2+s*q^2+s-t^2-t^4*s*q^3+t^2*s^2*q^3
        +t*s^2+t^2*s*q-t^3*s*q-t^3*s-t^3*q^3-t^2*s*q^2
        +t*s*q^3+t*s*q^2+2*t*s-t^3*q-t^3-2*t^3*s*q^3
        +t*s^2*q^3+t*s^2*q^2$
a11: t^2*s*q+t^2*s-t^2*s*q^3-t^2*s*q^2-s+t^2*q+t^2
        +t^4*s*q^3-t^2*s^2*q^3-t^2*s^2*q^2+t-t^3*s^2*q^3$
a12: t-t^3*s^2*q^3+1-t^4*s^2*q^3$
a12: (-1)*a12$
a20: t*s^2*q-t^2*s^2-t^3*q^-3+t^2*q^-2-t^2*s*q+t^3*s+t^2*s*q^-3
        -t*s*q^-2-t*s+t^3*s*q^-2$
a21: -t^2*s^2*q+t^3*s^2*q+t^3*s^2+t^2*q^-3-t*q^-2-t*q^-3
        +t^3*s*q-t*s*q^-3$
a22: t^3*s^2*q-t*q^-3$
a23: -t^4*s^2*q+q^-3$
p0: 1+t+t^2+t^4+t^5+t^6$
p1: -t-t^3-t^5$
p2: t^2+t^3+t^4$
p3: -t^3$
x0: 1$
x1: 0-a00*x0$
x1: x1/a01$
x2: a10*x0+a11*x1$
x2: 0-x2/a12$
x3: a20*x0+a21*x1+a22*x2$
x3: 0-x3/a23$
sum: p0*x0+p1*x1+p2*x2+p3*x3$
rhs: (1-q*t^3*s)*(1-q^2*t^3*s)*(1-t^2)*(1-q*t^2)$
rhs: rhs*(1-t^3*s^3)*(1-q^3*t^3)$
rhs: rhs/((1-q*t^2*s)*(1-q^2*t^2*s)*(1-t*s)*(1-q*t)*(1-t)*
        (1-q^3*t^3*s^2))$
sum: sum-rhs$
ratsimp(sum);
quit();
```

In the *input file* we solve for $x1$, $x2$, $x3$ successively. Then we ask MACSYMA to compute "sum" $= H_{a+1,b}/H_{a,b}$. We enter $R_{a+1,b}/R_{a,b}$ given in (9.4) and call it "rhs." So far every line has been terminated with a dollar sign so that the partial steps are not going to be printed out. The second line from the bottom is

sum: *sum* − *rhs*$

that defines the new "sum" to be the difference between the conjectured right-hand side and the real right-hand side. This should be zero if the conjecture is true. The last line asks MACSYMA to simplify this difference: *ratsimp*(*sum*); and now, finally, there is a semicolon, because now we *do* want to see the answer.

On December 22, 1986, 3:30 p.m., after two previous unsuccessful attempts (due to typing errors that were presently detected), I typed on my terminal:

macsyma < *inputfile*

After a few minutes came the output: 27 blank double lines (due to the dollar signs) and

$(c28)$
$(d28)$ 0

YEA!!!

□

10. Prospects. The next in line is F_4. But before we can do F_4, we must do D_4, the short part of F_4. A preliminary calculation done by Dave Robbins shows that for D_4, (4.6) involves more than one hundred terms. So we will have to find and solve a system of more than one hundred linear equations with rather complicated coefficients. While this is still within the reach of current computers, it is hard to justify that kind of expense before all other means have been exhausted.

As I have already mentioned, the reason why the Macdonald–Morris conjectures (qM-M2) are easier than the original Macdonald conjectures (qM), is that the two parameters let us break the problem into two subproblems. In a way we are first doing the long roots and only then the short roots. But nowhere in § 8 have we ever used the "physical appearance" of the short and long roots, that is the fact that the roots of R_{long} are "longer" than those of R_{short}. All we used was the fact that the partition $R = R_{\text{short}} \cup R_{\text{long}}$ partitions the root-system R into two subsets both of which are invariant under the action of the Weyl group W.

Is it possible to find such a partition for those root systems that have only one root length (A_n, D_n, E_6, E_7, E_8)? The answer is: not quite, but almost. Instead of the Weyl group W itself, we have to settle for invariance under a certain subgroup of W. It turns out that it is possible to find such a partition of R which is invariant under a very large subgroup of W, so we only have to sacrifice a little bit of symmetry. Still, we have to put up with some vectors that were previously denounced as bad guys. In return, however, the corresponding polynomial that appears in (4.1) is much smaller and the trade-off is well in our favor, since the resulting set S in (4.6) turns out to be much smaller.

For example, A_{n-1} can be partitioned into

$$A_{n-1} = \{\pm(e_1 - e_i);\, 2 \leqq i \leqq n\} \cup \{\pm(e_i - e_j);\, 2 \leqq i < j \leqq n\}.$$

The second set is the subroot system A_{n-2} in the last $n-1$ coordinates and its Weyl group S_{n-1} (that acts by permuting the last $n-1$ coordinates) is the subgroup that leaves both subsets invariant.

In fact, the first subset above can be further partitoned into its positive and negative roots and so A_{n-1} can be partitioned into *three* subsets, each of which is invariant under the above-mentioned S_{n-1}. Indeed, we have

$$A_{n-1} = \{e_1 - e_i;\, 2 \leqq i \leqq n\} \cup \{-e_1 + e_i;\, 2 \leqq i \leqq n\} \cup \{\pm(e_i - e_j;\, 2 \leqq i < j \leqq n)\}.$$

We should thus expect a three parameter "pseudo"-Macdonald–Morris conjecture

$$\text{(10.1)} \quad \text{C.T.} \prod_{i=2}^{n} (x_1/x_i)_a \prod_{i=2}^{n} (qx_i/x_1)_b \prod_{2 \leqq i < j \leqq n} (x_i/x_j)_c (qx_j/x_i)_{c-1} = \text{something explicit.}$$

Such an identity indeed exists and was conjectured by Morris [Mo] (Morris proved the $q = 1$ case). It was recently proved by Kadell [Kad1] and Habsieger [Hab2], who deduced it from their Askey q-Selberg integral. However it is possible to get a Stembridge-style proof by using the method of § 8 [Z4]. The $a = 0$ case is easily seen to be equivalent to the $a = b = 0$ case. Then one inducts on a and gets a recurrence in a. The analogue of (4.6) contains only n terms and it is easy to find $n-1$ independent equations satisfied by them. The base case $a = b = 0$ is just the A_{n-1} case while the $a = b = c$ is the A_n case. This provides the necessary induction.

For D_n the situation is not quite as rosy, but it is still very promising. While it is not possible to partition D_n into *three* subsets invariant under a large subgroup of W, it is possible to do it with *two* subsets.

Indeed,

$$D_n = \{\pm e_1 \pm e_i;\ 2 \leqq i \leqq n\} \cup \{\pm e_i + e_j;\ 2 \leqq i < j \leqq n\}. \tag{10.2}$$

The second set is just the root-system D_{n-1} on the last $n-1$ coordinates. The Weyl group of this D_{n-1} consists of all signed permutations with an even number of signs that act on the last $n-1$ coordinates [Bo, p. 257, (X)]. It leaves both subsets of (10.2) invariant. I conjecture that if

$$\begin{aligned} H_{a,b} =: \text{C.T.} \prod_{i=2}^{n} & (x_1/x_i)_a (qx_i/x_1)_{a-1} (x_1x_i)_a (q/x_1x_i)_{a-1} \\ & \cdot \prod_{2 \leqq i < j \leqq n} (x_i/x_j)_b (qx_j/x_i)_{b-1} (x_ix_j)_b (q/x_ix_j)_{b-1}, \end{aligned} \tag{10.3}$$

then $H_{a,b}$ has an explicit and perhaps nice expression. In any case the method described in § 8 should produce $H_{a,b+1}/H_{a,b}$, a certain rational function, and whether it is nice or not it should give us a formula for $H_{a,b}$ that we know should be nice when $a = b$. In any case the analogue of (4.6) is much simpler now, and the number of equations needed is considerably reduced. The base case $a = 1$ is essentially D_{n-1}, and once we obtain the recurrence in a, and thus the expression for $H_{a,b}$, then $H_{b,b}$ will give the D_n case of (qM). Once we will have D_n, the remaining classical families B_n and C_n should be relatively easy. D_n is the "hard core" of both B_n and C_n, and it hopefully would be relatively easy to add the rest. Similarly, it should be possible to find more refined conjectures for F_4 and the E's that will enable us to break the proof into manageable parts.

Another possibility is to find the "trivializing generalization": a much more general statement than the Macdonald conjectures that would be trivial (or at least easy, or in any case possible) to prove. Except for some coefficients in the A_{n-1} case [Ste3], the general coefficients of G_k and $G_{a,b}$ do not seem to have nice expressions. So we have to abandon the hope of finding a *nice* expression for the general coefficient of G_k. But perhaps it is possible to find certain *linear combinations* of these messy coefficients that are good-looking. Remember that in our method the desired coefficient, H_{k+1}, was obtained, via (4.6) as a certain linear combination of more or less ugly coefficients of the k case. Maybe it is possible to find a family of polynomials, a_λ, say, parametrized by partitions λ such that

$$\text{C.T.} \left[\prod_{\alpha \in R^+} (x^\alpha)_k (qx^{-\alpha})_{k-1} a_\lambda \right] \tag{10.4}$$

has a nice expression in k and λ. Now that we have a laboratory for producing not only the constant term, but also other coefficients of $F_k(F_{a,b})$, there is a vast hunting ground for formulating and testing such more general conjectures (see [Kad2] for a similar idea in the context of the Selberg integral).

A related idea, inspired by Aomoto's [Ao] proof of Selberg's integral, was suggested by Askey [As3]: Break the ascent from k to $k+1$ in (qM) by raising the subscripts on the roots one, or few, at a time. The present method also offers a convenient workbench for Askey's approach. In particular it is possible to verify his G_2 conjectures made at the end of § 4 of [As3].

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Note added in proof. Kevin Kadell has meanwhile proved the BC_n cases of (qM-M1), and thus also the B_n, C_n, D_n cases. Frank Garvan (preprint) has used the method of this paper to prove the $q=1$ case of F_4. He also succeeded in proving the F_4, I_3 cases of the Macdonald-Mehta conjectures. Frank Garvan and Dennis Stanton have proved that the system (5.9) is always upper triangular, in the $q=1$ case.

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