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A q-Foata Proof of the q-Saalschütz Identity

DORON ZEILBERGER

Foata's beautiful proof of the Saalschutz identity is q-iefied.

Dominique Foata [2] [6] gave a beautiful combinatorial proof of the following binomial coefficients identity, that is trivially equivalent to the famous Pfaff–Saalschutz identity:

$$\binom{a+b}{a+k}\binom{a+c}{c+k}\binom{b+c}{b+k} = \sum_n \frac{(a+b+c-n)!}{(a-n)!\,(b-n)!\,(c-n)!\,(n+k)!\,(n-k)!}. \tag{1}$$

Here we show how to q-ify his argument, thus proving Jackson's [3] q-analog of Saalschutz.

$$\begin{bmatrix} a+b \\ a+k \end{bmatrix}\begin{bmatrix} a+c \\ c+k \end{bmatrix}\begin{bmatrix} b+c \\ b+k \end{bmatrix} = \sum_n q^{n^2-k^2} \frac{[a+b+c-n]!}{[a-n]!\,[b-n]!\,[c-n]!\,[n+k]!\,[n-k]!}. \tag{1q}$$

Here $[m]! = 1\cdot(1+q)(1+q+q^2)\ldots(1+q+\cdots+q^{m-1})$. Other combinatorial proofs of q-Saalschutz have recently been given by Andrews and Bressoud [1] and by Goulden [5].

Assume that you have a finite alphabet that has been totally ordered. An inversion in a word $a_1, a_2, \ldots, a_k$ is a pair of indices i, j such that $i < j$ but $a_i > a_j$. We denote by $\mathrm{inv}(w)$ the number of inversions of w. For example $\mathrm{inv}(3121332) = 7$. It is well known ([4], 5.1) that if $M(a_1, \ldots, a_r)$ is the set of words that have a_1 1's, $\ldots$, a_r r's then

$$\sum_{w\in M(a_1,\ldots,a_r)} q^{\mathrm{inv}(w)} = \frac{[a_1 + \cdots + a_r]!}{[a_1]!\ldots[a_r]!}. \tag{2}$$

Now consider the set $M_{a,b,c,k} = M(a+k, b-k)\times M(a-k, c+k)\times M(b+k, c-k)$ consisting of triplets of words (w_1, w_2, w_3) where

$$\begin{aligned} &w_1 \text{ has } a+k \text{ 1's, } b-k \text{ 2's,}\\ &w_2 \text{ has } a-k \text{ 1's, } c+k \text{ 2's,}\\ &w_3 \text{ has } b+k \text{ 2's, } c-k \text{ 3's,}\end{aligned}$$

and define

$$\text{weight}\,(w_1, w_2, w_3) = q^{\mathrm{inv}(w_1)+\mathrm{inv}(w_2)+\mathrm{inv}(w_3)}$$

then, of course, because of (2)

$$\text{weight}\,(M_{a,b,c,k}) = \begin{bmatrix} a+b \\ a+k \end{bmatrix}\begin{bmatrix} a+c \\ c+k \end{bmatrix}\begin{bmatrix} b+c \\ b+k \end{bmatrix}.$$

We will now describe Foata's proof in a form that makes it q-generalizable. Let's present first a typical element of $M_{a,b,c,k}$ say with $a = 3$, $b = 4$, $c = 3$, $k = 1$ where w_1, w_2, w_3 are

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Here we show how to q-ify his argument, thus proving Jackson's [3] q-analog of Saalschutz.

$$\begin{bmatrix} a+b \\ a+k \end{bmatrix}\begin{bmatrix} a+c \\ c+k \end{bmatrix}\begin{bmatrix} b+c \\ b+k \end{bmatrix} = \sum_n q^{n^2-k^2} \frac{[a+b+c-n]!}{[a-n]!\,[b-n]!\,[c-n]!\,[n+k]!\,[n-k]!}. \quad (1q)$$

Here $[m]! = 1\cdot(1+q)(1+q+q^2)\ldots(1+q+\cdots+q^{m-1})$. Other combinatorial proofs of q-Saalschutz have recently been given by Andrews and Bressoud [1] and by Goulden [5].

Assume that you have a finite alphabet that has been totally ordered. An inversion in a word $a_1, a_2, \ldots, a_k$ is a pair of indices i, j such that $i < j$ but $a_i > a_j$. We denote by $\mathrm{inv}(w)$ the number of inversions of w. For example $\mathrm{inv}(3121332) = 7$. It is well known ([4], 5.1) that if $M(a_1, \ldots, a_r)$ is the set of words that have a_1 1's, $\ldots$, a_r r's then

$$\sum_{w\in M(a_1,\ldots,a_r)} q^{\mathrm{inv}(w)} = \frac{[a_1+\cdots+a_r]!}{[a_1]!\ldots[a_r]!}. \quad (2)$$

Now consider the set $M_{a,b,c,k} = M(a+k, b-k) \times M(a-k, c+k) \times M(b+k, c-k)$ consisting of triplets of words (w_1, w_2, w_3) where

w_1 has $a+k$ 1's, $b-k$ 2's,

w_2 has $a-k$ 1's, $c+k$ 2's,

w_3 has $b+k$ 2's, $c-k$ 3's,

and define

$$\text{weight}\,(w_1, w_2, w_3) = q^{\mathrm{inv}(w_1)+\mathrm{inv}(w_2)+\mathrm{inv}(w_3)}$$

then, of course, because of (2)

$$\text{weight}\,(M_{a,b,c,k}) = \begin{bmatrix} a+b \\ a+k \end{bmatrix}\begin{bmatrix} a+c \\ c+k \end{bmatrix}\begin{bmatrix} b+c \\ b+k \end{bmatrix}.$$

We will now describe Foata's proof in a form that makes it q-generalizable. Let's present first a typical element of $M_{a,b,c,k}$ say with $a = 3, b = 4, c = 3, k = 1$ where w_1, w_2, w_3 are

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written in row 1, row 2, and row 3, respectively. For example

$$\begin{matrix} 1&2&1&2&2&1&1& \\ 1&3&3&1&3&3& & \\ 3&2&2&2&3&2&2& \end{matrix}, \tag{3E}$$

whose weight is $q^7 \cdot q^2 \cdot q^7 = q^{16}$.

We will now read our triplets of words 'in Chinese' that is by columns and make from a triplet of words just one word 'in Chinese' using the 'Chinese alphabet' of 5 letters

$$\begin{pmatrix}1\\1\\-\end{pmatrix}\begin{pmatrix}1\\3\\2\end{pmatrix}\begin{pmatrix}2\\-\\2\end{pmatrix}\begin{pmatrix}2\\1\\3\end{pmatrix}\begin{pmatrix}-\\3\\3\end{pmatrix}.$$

If the first column is $\left(\begin{smallmatrix}1\\1\\2\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}1\\1\\3\end{smallmatrix}\right)$ we write down $\left(\begin{smallmatrix}1\\1\\ \end{smallmatrix}\right)$, cross out the 1s in the first and second rows and keep going. Similarly if the first column is either $\left(\begin{smallmatrix}2\\1\\2\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}2\\3\\2\end{smallmatrix}\right)$ we write down $\left(\begin{smallmatrix}2\\ \\2\end{smallmatrix}\right)$, cross out the 2's in the first and third rows and keep going. Similarly for $\left(\begin{smallmatrix}1\\3\\3\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}2\\3\\3\end{smallmatrix}\right)$. If you have a column with $\left(\begin{smallmatrix}1\\3\\2\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}2\\1\\3\end{smallmatrix}\right)$ you just take them.

For example, here is how to read in Chinese the above triplet (3E)

$$\begin{pmatrix}1\\1\\-\end{pmatrix}\begin{pmatrix}-\\3\\3\end{pmatrix}\begin{pmatrix}2\\-\\2\end{pmatrix}\begin{pmatrix}1\\3\\2\end{pmatrix}\begin{pmatrix}2\\-\\2\end{pmatrix}\begin{pmatrix}2\\1\\3\end{pmatrix}\begin{pmatrix}1\\3\\2\end{pmatrix}\begin{pmatrix}1\\3\\2\end{pmatrix}. \tag{3C}$$

It is easy to see that this can always be done. How many times does every 'Chinese letter' appear? Let there be

$$e_1\begin{pmatrix}1\\1\\-\end{pmatrix}, e_2\begin{pmatrix}1\\3\\2\end{pmatrix}, e_3\begin{pmatrix}2\\-\\2\end{pmatrix}, e_4\begin{pmatrix}2\\1\\3\end{pmatrix}, e_5\begin{pmatrix}-\\3\\3\end{pmatrix},$$

then by counting the letters of w_1, w_2, and w_3

$$\begin{aligned} e_1 + e_2 &= a + k, & e_3 + e_4 &= b - k, \\ e_1 + e_4 &= a - k, & e_2 + e_5 &= c + k, \\ e_2 + e_3 &= b + k, & e_4 + e_5 &= c - k. \end{aligned}$$

This system of linear equations has the following solution (n being an integer parameter) $e_1 = a - n$, $e_2 = n + k$, $e_3 = b - n$, $e_4 = n - k$, $e_5 = c - n$.

Conversely every word in the 'Chinese' alphabet with the above specifications gives rise to a triplet in $M_{a,b,c,k}$. Thus there is a bijection

$$\text{(F)}\ M_{a,b,c,k} \leftrightarrow \bigcup_n M(a - n, n + k, b - n, n - k, c - n)$$

and taking cardinalities yields (1). This is Foata's proof. How can it be q-ified?

We have to introduce weight on the set of Chinese words $M(e_1, e_2, e_3, e_4, e_5)$ in the letters $\left(\begin{smallmatrix}1\\1\\-\end{smallmatrix}\right) \ldots \left(\begin{smallmatrix}-\\3\\3\end{smallmatrix}\right)$. If we are going to use inversions we will have to order them. Let's try

$$\begin{pmatrix}1\\1\\-\end{pmatrix} < \begin{pmatrix}1\\3\\2\end{pmatrix} < \begin{pmatrix}2\\-\\2\end{pmatrix} < \begin{pmatrix}2\\1\\3\end{pmatrix} < \begin{pmatrix}-\\3\\3\end{pmatrix}$$

and let us see if the number of inversions is preserved in the translation from 'Chinese' to 'English'.

$\begin{pmatrix}1\\1\end{pmatrix}$ in front of $\begin{pmatrix}1\\3\\2\end{pmatrix}$ does not cause any inversions neither in Chinese nor in English. $\begin{pmatrix}1\\3\\2\end{pmatrix}$ in front of $\begin{pmatrix}1\\1\end{pmatrix}$ causes one inversion in Chinese and one inversion (31) in English. Doing this for all the $\binom{5}{2} = 10$ pairs of Chinese letters shows that everything is fine *except* the pair $\begin{pmatrix}1\\3\\2\end{pmatrix}$ and $\begin{pmatrix}2\\1\\3\end{pmatrix}$.

$\begin{pmatrix}1\\3\\2\end{pmatrix}$ in front of $\begin{pmatrix}2\\1\\3\end{pmatrix}$ does not cause any inversions in Chinese but one inversion (31) in English. $\begin{pmatrix}2\\1\\3\end{pmatrix}$ in front of $\begin{pmatrix}1\\3\\2\end{pmatrix}$ causes one inversion in Chinese and 2 in English. So in either case there is one inversion too many in the English translation for every one of the e_2e_4 pairs of $\begin{pmatrix}1\\3\\2\end{pmatrix}$ and $\begin{pmatrix}2\\1\\3\end{pmatrix}$ that occur in every word of $M(e_1, e_2, e_3, e_4, e_5)$. It follows that the weight in $M(e_1, e_2, e_3, e_4, e_5)$ should be defined by

$$\text{weight}\ (W) \ = \ q^{e_2e_4}q^{\mathrm{inv}(W)}.$$

We have just shown that with this modified weight Foata's bijection (F) (i.e. the 'translation' from 'English' to 'Chinese') is weight preserving, and thus taking the weights of both sides of (F) and using (2) yields the q-Saalchütz identity $(1q)$.

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