One-Line Proofs of the Unimodality of $\left[{n \atop 3}\right]$ and $\left[{n \atop 4}\right]$

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A polynomial of degree A: $a_0+a_1q+\ldots+a_Aq^A$ is *symmetric* and *unimodal* if $a_i = a_{A-i}$ and $a_0 \le a_1 \le \cdots \le a_{[A/2]} \ge \cdots \ge a_A$.
If $a_A = 0$ we will say that the polynomial is "viewed as a polynomial of degree A". (For example q^2 is not unimodal when viewed as a polynomial of degree 2 but is unimodal when viewed as a polynomial of degree 4).

The Gaussian polynomials $\left[{n \atop m}\right]$ are defined by

$$\left[{n \atop m}\right] = \frac{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-m+1})}{(1-q)\ldots(1-q^m)} \tag{1}$$

and are of degree $m(n-m)$. It is a consequence of a deep theorem of Sylvester [3] that they are unimodal, for every n and m, but no elementary proof is known. For $m = 3,4$ elementary proofs do exist and are due to Riess [2] ($m = 3$ and 4), Lindstrom [1] ($m = 3$) and West [4] ($m = 4$). Neither proof is really short, although, in fairness, they prove something stronger.

Based on an elementary remark and easily verified identities, we can give one-line proofs of the unimodality of the Gaussian polynomials (1) for $m = 2,3,4$.

Remark. If the polynomial $P(q)$ is symmetric and unimodal when viewed as a polynomial of degree A then for any integer B, $q^BP(q)$ is unimodal when viewed as a polynomial of degree $A + 2B$.

Lemma. *The following recursive identities hold*

$$\left[{n \atop 2}\right] = q^2\left[{n-2 \atop 2}\right] + (1+q+\ldots+q^{2(n-2)}) \tag{2}$$

$$\left[{n \atop 3}\right] = q^6\left[{n-4 \atop 3}\right] + \sum_{\substack{i=0 \\ i\neq 1}}^{n-3}(q^i+\ldots+q^{3(n-3)-i}) \tag{3}$$

$$S(n) = q^6S(n-3) + \sum_{\substack{i=0 \\ i\neq 1}}^{n-4}(q^i+\ldots+q^{4(n-4)-i}), \tag{4}$$

where

$$S(n) = \begin{bmatrix} n \\ 4 \end{bmatrix} - q^4 \begin{bmatrix} n-2 \\ 4 \end{bmatrix}.$$

Proof. By summing all geometric series and substituting the definition (1) it is readily seen that all quantities are polynomials in $x = q^n$ with coefficients that are rational functions of q. Specifically equation (m) involves polynomials of degree m $(m = 2,3,4)$. Since two polynomials of degree r coincide if and only if they coincide for $r + 1$ values it is enough to check (m) for the $m + 1$ values $n = m, \ldots, 2m$. □

Theorem. $\begin{bmatrix} n \\ m \end{bmatrix}$ *is symmetric and unimodal for* $m = 2,3,4$.

Proof. For $m = 2,3,4$ this follows by induction on n, form the Remark, and the identity (m); for $m = 4$ we obtain first that $S(n)$ is symmetric and unimodal and then we apply the Remark and induction once again to

$$\begin{bmatrix} n \\ 4 \end{bmatrix} = q^4 \begin{bmatrix} n-2 \\ 4 \end{bmatrix} + S(n). \quad \square$$

Acknowledgements. Many thanks are due to the referee, whose valuable suggestions considerably improved the readability of this note. This research was partly sponsored by the NSF.

References.

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