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A PROOF OF ANDREWS' q-DYSON CONJECTURE

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Let $(y)_a=(1-y)(1-qy)\cdots(1-q^{a-1}y)$. We prove that the constant term of the Laurent polynomial $\prod_{1\le i<j\le n}(x_i/x_j)_{a_i}(qx_j/x_i)_{a_j}$, where $x_1,\ldots,x_n$, q are commmuting indeterminates and $a_1,\ldots,a_n$ are non-negative integers, equals $(q)_{a_1+\cdots+a_n}/(q)_{a_1}\cdots(q)_{a_n}$. This settles in the affirmative a conjecture of George Andrews (in: R.A. Askey, ed., Theory and Applications of Special Functions, Academic Press, New York, 1975, 191–224].

Introduction

In 1962, Dyson [3] made the following conjecture:

$$\text{the constant term of } \prod_{1\le i\ne j\le n}\left(1-\frac{x_i}{x_j}\right)^{a_i} \text{ is equal to } (a_1+\cdots+a_n)!/a_1!\cdots a_n! \tag{D}$$

This conjecture was settled by Gunson [8] and Wilson [14] and in 1970, Good [7] gave a short and very elegant proof.

In 1975, Andrews [1] conjectured the following q-analog:

let $(y)_a=(1-y)(1-qy)(1-q^2y)\cdots(1-q^{a-1}y)$, $(y)_0=1$, $(y)_{-1}=(1-yq^{-1})^{-1}$, then

$$\text{the constant term of } \prod_{1\le i<j\le n}\left(\frac{x_i}{x_j}\right)_{a_i}\left(\frac{qx_j}{x_i}\right)_{a_j} \text{ is equal to } (q)_{a_1+\cdots+a_n}/(q)_{a_1}\cdots(q)_{a_n}. \tag{A}$$

Andrews' conjecture generalizes Dyson's since the latter is the case $q=1$ of the former.

An excellent exposition of Andrews' conjecture, as well as of some related conjectures of Macdonald, is given in Morris' [13] thesis. Morris writes: "Independent proofs of Andrews' conjecture for $n>3\ldots$ would provide many deep

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examples of multiple basic and ordinary hypergeometric identities, a topic about which little is currently known."

One natural way to try to prove this conjecture was to try to emulate Good and find a difference-equation proof. This is essentially the approach taken by Kadell [9] in his proof of (A) for $n=4$. It has not been successful for larger values of n, but the attempt did lead one of us (D.Z.) to a general theory of hypergeometric sums [15].

Another line of attack, which led to D.Z.'s combinatorial proof of Dyson's conjecture [16], was to try to employ the beautiful ideas of Foata [4]. This approach failed as well. What finally did work was a synthesis of the Good (difference-equation) approach and of the Foata (combinatorial) approach. If it were not for their ideas this proof would never have come to be. We also benefited from a clever idea of Gessel [6].

We will prove Andrews' conjecture (A) by proving an equivalent identity, namely

$$\sum_{K\in\mathcal{K}} f(K) \prod_{1\le i\ne j\le n} \frac{1}{(q)_{a_i+k_{ij}}} = \frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}} \prod_{1\le i<j\le n} \frac{1}{(q)_{a_i+a_j}} \tag{Z}$$

where $\mathcal{K}$ is the set of all matrices $K=(k_{ij})_{1\le i,j\le n}$ satisfying $k_{ji}=-k_{ij}$ and, for every i, $\sum_{j=1}^n k_{ij}=0$, and $f(K)$ is defined by

$$f(K)=(-1)^{\sum k_{ij}}q^{\sum k_{ij}(k_{ij}+1)/2},$$

both summations being over all pairs (i,j) for which $1\le i<j\le n$.

We shall end the introduction by showing that (A) and (Z) are equivalent.

An immediate consequence of the q-binomial theorem [10, 1.2.6. ex. 58] is the identity

$$(y)_a(qy^{-1})_b=\sum_k \frac{(-1)^k q^{k(k+1)/2}(q)_{a+b}}{(q)_{a+k}(q)_{b-k}}\, y^{-k}$$

where the summation is taken over all k, $-\infty<k<+\infty$, but $(q)_a^{-1}$ is defined to be zero for negative integral values of a. It follows that for each pair (i,j) such that $1\le i<j\le n$ we have the identity

$$\left(\frac{x_i}{x_j}\right)_{a_i}\left(\frac{qx_j}{x_i}\right)_{a_j}=\sum_{k_{ij}} \frac{(-1)^{k_{ij}}q^{k_{ij}(k_{ij}+1)/2}(q)_{a_i+a_j}}{(q)_{a_i+k_{ij}}(q)_{a_j+k_{ji}}}\, x_i^{-k_{ij}}x_j^{-k_{ji}}$$

where we have put $k_{ji}=-k_{ij}$.

Multiplying all these $\binom{n}{2}$ identities together and looking for the constant term shows that Andrews' conjecture is equivalent to

$$\sum_{K\in\mathcal{K}} (-1)^{\sum_{1\le i<j\le n}k_{ij}}q^{\sum_{1\le i<j\le n}k_{ij}(k_{ij}+1)/2} \prod_{1\le i<j\le n} \frac{(q)_{a_i+a_j}}{(q)_{a_i+k_{ij}}(q)_{a_j+k_{ji}}} = \frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}},$$

where the sum is over all $K=(k_{ij})\in\mathcal{K}$. Dividing through by $\prod_{1\le i<j\le n}(q)_{a_i+a_j}$ yields (Z).

1. Combinatorial preliminaries

A *partition* p with m parts is a non-increasing sequence of m non-negative numbers. That is $p: p^{(1)} \geqslant \cdots \geqslant p^{(m)} \geqslant 0$. Unlike common usage [2] we allow zeros so what we call "a partition with m parts" can be trivially identified, by chopping the zeros, with what is known in common parlance as "a partition with at most m parts". The number of parts of a partition is denoted by $\#p$ and the sum of its elements by $|p|$. Thus, for example, $\#(332100)=6$, $\#(32210)=5$, $\#(3321)=4$, $|332100|=|33210|=|3321|=9$.

The weight of a partition p is defined by $\text{weight}(p)=q^{|p|}$. Thus $\text{weight}(32110)=q^7$, $\text{weight}(00000)=1$. Given any set $\mathscr{A}$ on whose elements there is a weight defined, we denote by $\text{weight}(\mathscr{A})$ the sum of all the weights of the individual elements: $\text{weight}(\mathscr{A})=\sum_{A\in\mathscr{A}} \text{weight}(A)$. It is a known fact [11, 5.1.1 ex. 15] that the weight of the set of partitions with m parts is $1/(q)_m$.

Given sets $\mathscr{A}_1, \ldots, \mathscr{A}_N$ if we define a weight on the product $\mathscr{A}_1 \times \cdots \times \mathscr{A}_N$ by $\text{weight}(A_1, \ldots, A_N)=\text{weight}(A_1)\cdot \ldots \cdot \text{weight}(A_N)$, then, of course, $\text{weight}(\mathscr{A}_1 \times \cdots \times \mathscr{A}_N)=\text{weight}(\mathscr{A}_1) \cdots \text{weight}(\mathscr{A}_N)$.

We will have occasion to consider creatures called partition matrices, which are matrices $P=(p_{ij})_{1\leqslant i,j\leqslant n}$ whose entries p_{ij} are partitions. The weight is defined by

$$\text{weight}(P)=q ** \Big(\sum_{1\leqslant i,j\leqslant n} |p_{ij}|\Big),$$

where $q ** x$ denotes q^x. Given a numerical matrix $(c_{ij})_{1\leqslant i,j\leqslant n}$ it is obvious that the weight of the set of partition matrices $P=(p_{ij})_{1\leqslant i,j\leqslant n}$ having $\#p_{ij}=c_{ij}$ is

$$\prod_{1\leqslant i,j\leqslant n} \frac{1}{(q)_{c_{ij}}}.$$

Another important combinatorial species is the *word.* A word in the alphabet $\{1, \ldots, n\}$ of type $1^{a_1}2^{a_2} \ldots n^{a_n}$ is any sequence containing exactly a_1 1's, a_2 2's, $\ldots$, a_n n's. We will denote the set of words of type $1^{a_1} \ldots n^{a_n}$ by $M(a_1, \ldots, a_n)$. For example, the members of $M(1,2,1)$ are {1223, 1232, 1322, 2123, 2132, 2213, 2231, 2312, 2321, 3122, 3212, 3221}.

Every word W on n letters gives rise to $\binom{n}{2}$ 2-lettered words $(W_{ij})_{1\leqslant i<j\leqslant n}$ where W_{ij} is the word with the letters i and j, of type $i^{a_i}j^{a_j}$ obtained by retaining only the letters i and j. For example if $W=41211321133214\in M(6,3,3,2)$ then

$$\begin{array}{lll} W_{12}=121121121 & W_{13}=111311331 & W_{14}=41111114 \\ & W_{23}=232332 & W_{24}=42224 \\ & & W_{34}=43334. \end{array}$$

This paper would have been much harder to write were it not for the useful χ notation, popularized by Adriano Garsia. For any statement A we write $\chi(A)=1$ if A is true and $\chi(A)=0$ if A is false. For example $\chi(1+1=3)=0$, $\chi(2+3=5)=1$.

The major index of a word $W=W_1\ldots W_l$ is defined as $\operatorname{maj}(W)=\sum_{i=1}^{l-1} i\chi(W_i>W_{i+1})$. This notion was introduced by MacMahon (see [1]) who proved that

$$\sum_{W\in M(a_1,\ldots,a_n)} q^{\operatorname{maj}(W)}=\frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}}. \tag{1.1}$$

A related notion, which in this paper is only used for permutations, is that of the number of inversions $\operatorname{inv}(W)=\sum_{1\le\alpha<\beta\le l}\chi(W_\alpha>W_\beta)$. It is immediate that $\operatorname{inv}(W)=\sum_{1\le i<j\le n}\operatorname{inv}(W_{ij})$, on the other hand it is grossly wrong that $\operatorname{maj}(W)=\sum_{1\le i<j\le n}\operatorname{maj}(W_{ij})$. However $\sum_{1\le i<j\le n}\operatorname{maj}(W_{ij})$ can be used to define a brand-new statistic, which for lack of a better name, we will call the z-index: $z(W)=\sum_{1\le i<j\le n}\operatorname{maj}(W_{ij})$. It is a well-known fact that

$$\sum_{W\in M(a_1,\ldots,a_n)} q^{\operatorname{inv}(W)}=\frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}}$$

and thus

$$\sum_{W\in M(a_1,\ldots,a_n)} q^{\operatorname{maj}(W)}=\sum_{W\in M(a_1,\ldots,a_n)} q^{\operatorname{inv}(W)}.$$

Foata ([5], see also [11, 5.1.1 ex. 19]) gave a beautiful bijective proof of this identity. One of the cornerstones of the present paper is (Lemma 4.1)

$$\sum_{W\in M(a_1,\ldots,a_n)} q^{z(W)}=\frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}},$$

but our proof is by induction (the kind of proof G.H. Hardy used to call "essentially verification"). It would be nice to find a Foata-style proof.

The following Bijection M is *crucial* for Section 3. It occurs in MacMahon's (see [11, p. 18]) proof of (1.1).

Bijection M. Let $W=W_1\cdots W_l$ be a prescribed word on the alphabet $\{1,\ldots,n\}$. There is a bijection between partitions $p:p^{(1)}\ldots p^{(l)}$ satisfying $p^{(i)}>p^{(i+1)}$ whenever $W_i>W_{i+1}$ and ordinary partitions $q:q^{(1)}\ldots q^{(l)}$ such that

$$|p|=|q|+\operatorname{maj}(W).$$

Description. Scan W from left to right. Whenever you encounter a descent, that is, an i with $W_i>W_{i+1}$, you know that $p^{(i)}>p^{(i+1)}$, change

$$(p^{(1)}\ldots p^{(i)}p^{(i+1)}\ldots p^{(l)})\leftarrow((p^{(1)}-1)\ldots(p^{(i)}-1)p^{(i+1)}\ldots p^{(l)});$$

keep doing it until you have finished scanning W. The final outcome is q.

Example. $p:333222111$; $W=122122112$; the third place is a descent so $p\leftarrow 222222111$; the next descent is the sixth place so $p\leftarrow 111111111$ and since there are no more descents, $q=111111111$.

The last notion which we are going to use is that of the *tournament.* A tournament on n players $\{1, \ldots, n\}$ is a skew-symmetric matrix $(t_{ij})_{1\leq i\neq j\leq n}$, $t_{ij} = t_{ji}$, such that $t_{ij} = i$ or j. If $t_{ij} = i$ we say "i beats j" and if $t_{ij} = j$ we say "j beats i". A tournament is called *transitive* if for $1 \leq i \neq j \neq k \leq n$, "$i$ beats j" and "j beats k" implies "i beats k". Otherwise it is *non-transitive.* For example

$$\begin{matrix} t_{12}=1 & t_{13}=3 \\ & t_{23}=3 \end{matrix} \quad \text{is transitive}$$

while

$$\begin{matrix} t_{12}=1 & t_{13}=3 \\ & t_{23}=2 \end{matrix} \quad \text{is non-transitive.}$$

There are altogether $2^{\binom{n}{2}}$ tournaments, $n!$ of which are transitive. This is so because every transitive tournament defines a permutation as follows:

There is a player $\pi(1)$ who beat everybody else, a player $\pi(2)$ who beat everybody but $\pi(1), \ldots$ and finally a player $\pi(n)$ who got beaten by all. Given a permutation π, we will denote by (π_{ij}) the corresponding transitive tournament. (*Warning*: it should not be confused with the previous notation W_{ij}). Thus if $\pi = 2143$

$$\begin{matrix} \pi_{12}=2 & \pi_{13}=1 & \pi_{14}=1 \\ & \pi_{23}=2 & \pi_{24}=2 \\ & & \pi_{34}=4. \end{matrix}$$

A cycle in a non-transitive tournament is a sequence $(i_1, i_2, \ldots, i_k)$ such that i_1 beats i_2, i_2 beats $i_3, \ldots, i_{k-1}$ beats i_k and i_k beats i_1. If there exists a single player who is contained in every cycle, he is called a *spoiler* for the tournament. Note that removing a spoiler from a tournament breaks all cycles and so makes it transitive. Of course, not every non-transitive tournament has a spoiler, and if it does then this element is not necessarily unique. For example, in

$$\begin{matrix} t_{12}=1 & t_{13}=3 \\ & t_{23}=2 \end{matrix}$$

every element is a spoiler.

The *score vector* $\bar{w} = (w_1, \ldots, w_n)$ for a tournament is a record of how many games each player wins:

$$w_k = \sum_{1\leq i<k} \chi(t_{ik} = k) + \sum_{k<j\leq n} \chi(t_{kj} = k).$$

NonTrans$(n; \bar{w}; r)$ denotes the set of non-transitive tournaments with n players and score vector $\bar{w}$ and for which r is a spoiler.

We highly recommend that the reader look up Gessel's paper [6] which inspired much of this work.

2. The combinatorial interpretation and outline of the proof

Let $\mathcal{P} = \mathcal{P}(a_1, \ldots, a_n)$ be the set of partition matrices $(p_{ij})_{1\le i,j\le n}$ such that

(i) $\sum_{j=1}^{n} \#p_{ij} = (n-1)a_i \ (i = 1, \ldots, n)$,

(ii) $\#p_{ij} + \#p_{ji} = a_i + a_j \ (1 \le i < j \le n)$,

(iii) $\#p_{ii} = 0$, that is the diagonal entries are empty.

Every such partition matrix defines uniquely a numerical matrix $K = (k_{ij})_{1\le i,j\le n}$ belonging to $\mathcal{K}$ (defined in the Introduction) where for $i \ne j$ $\#p_{ij} = a_i + k_{ij}$. For example the following is a member of $\mathcal{P}(4, 6, 3)$ with $k_{12} = 2$, $k_{13} = -2$, $k_{23} = 2$:

$$\begin{pmatrix} * & p_{12} = 333222 & p_{13} = 44 \\ p_{21} = 4310 & * & p_{23} = 22210000 \\ p_{31} = 44300 & p_{32} = 0 & * \end{pmatrix}.$$

We define a *weight* on $\mathcal{P}$ as follows:

$$\text{weight}(P) = f(K)q ** \left[\sum_{1\le i\ne j\le n} |p_{ij}| \right],$$

where K is the numerical matrix defined by P and $f(K)$ is as defined in the Introduction. For example, the weight of the partition matrix given above is

$$(-1)^{2-2+2} q^{2\cdot 3/2 + (-2)(-1)/2 + 2\cdot 3/2} q ** [|333222| \\ + |44| + |4310| + |22210000| + |44300| + |0|] = q^{56}.$$

For any given $K \in \mathcal{K}$ let $\mathcal{P}_K$ denote the subset of $\mathcal{P}$ having $\#p_{ij} = a_i + k_{ij}$ for $i \ne j$. By the remarks of Section 1:

$$\text{weight}(\mathcal{P}_K) = f(K) \prod_{1\le i\ne j\le n} \frac{1}{(q)_{a_i + k_{ij}}}.$$

Since $\mathcal{P} = \bigcup_{K\in\mathcal{K}} \mathcal{P}_K$ it follows that weight$(\mathcal{P})$ = l.h.s. of (Z).

In Section 3 we will introduce two sets $\mathcal{G} = \mathcal{G}(a_1, \ldots, a_n)$ and $\mathcal{B} = \mathcal{B}(a_1, \ldots, a_n)$ which we will name the *good guys* and the *bad guys* respectively. We will introduce appropriate weights on these sets and will prove

Theorem 3. *There is a weight preserving bijection between $\mathcal{P}$ and $\mathcal{B} \cup \mathcal{G}$.*

From this it follows that weight$(\mathcal{P})$ = weight$(\mathcal{G})$ + weight$(\mathcal{B})$. In Section 4 we will prove

Theorem 4. weight$(\mathcal{G})$ = *r.h.s. of* (Z).

In Section 5 we will prove

Theorem 5. weight$(\mathcal{B}) = 0$.

Combining all these would yield

$$\text{l.h.s. of (Z)} = \text{weight}(\mathcal{P}) = \text{weight}(\mathcal{G}) + \text{weight}(\mathcal{B})$$
$$= \text{r.h.s. of (Z)} + 0 = \text{r.h.s. of (Z)}.$$

3. The good guys and the bad guys

We shall begin by trying to motivate what separates the good guys from the bad guys. As we have seen in Section 2, the left-hand side of (Z) is the generating function for certain matrices of partitions. Scrutinizing the right-hand side of (Z) we see that a piece of it, namely

$$\prod_{1\leq i<j\leq n} \frac{1}{(q)_{a_i+a_j}}$$

is also the generating function for certain matrices of partitions, specifically for upper triangular matrices

$$Q = (Q_{ij})_{1\leq i<j\leq n}$$

where Q_{ij} is a partition with $a_i + a_j$ parts and the weight of Q is simply given by

$$q ** \Bigg[\sum_{1\leq i<j\leq n} |Q_{ij}|\Bigg].$$

This suggest that we want to transform matrices in $\mathcal{P}$ into the upper triangular matrices generated on the right-hand side. When we observe that for $i \neq j$:

$$\#p_{ij} + \#p_{ji} = a_i + k_{ij} + a_j + k_{ji} = a_i + a_j,$$

it is natural to transform a matrix P of $\mathcal{P}$ by dropping the empty partitions on the diagonal and then for each pair (i, j), $1 \leq i < j \leq n$, combining the parts of p_{ij} with those of p_{ji} to form $\bar{Q}_{ij}$. Thus

$$P = \begin{pmatrix} * & 531 & 320 \\ 32 & * & 41 \\ 21 & 20 & * \end{pmatrix}$$

would become

$$\bar{Q} = \begin{pmatrix} * & 53321 & 32210 \\ & * & 4210 \\ & & * \end{pmatrix}.$$

Of course, under this transformation there are many different matrices P which give rise to the same $\bar{Q}$. What we shall do is to accompany $\bar{Q}$ with a code which tells us how to decompose $\bar{Q}$ back to the appropriate P. This code will be a word $W \in M(a_1, \ldots, a_n)$ which is read as follows: for each pair (i, j), $1 \leq i < j \leq n$ and for each k, $1 \leq k \leq a_i + a_j$, the kth part of $\bar{Q}_{ij}$, namely $\bar{Q}_{i,j}^{(k)}$, comes from the partition p_{ij} if and only if the kth letter of W_{ij} is i.

For the example given above, the word 1123231 is the code for decomposing $\bar{Q}$ to get back P. For example, $W_{23}=2323$; $\bar{Q}_{23}=4210$ and thus $p_{23}=41$ while $p_{32}=20$.

It should be obvious from the definition of the decomposition procedure that P can be decomposed from a Q using a code word only if $P\in\mathcal{P}_K$ where K is the zero matrix (i.e. $\#p_{ij}=a_i$ for all $1\leqslant i\neq j\leqslant n$). As we shall see, this is not a sufficient condition. If P is in $\mathcal{P}_K$, K the zero matrix, then we attempt to construct the code word W as follows:

Algorithm 3.1

Step 1. Initialize W to be the empty word, $(B_{ij})=(p_{ij})$.

Step 2. If any of the partitions, say B_{ij}, $i\neq j$, is empty, then all partitions in the ith row are empty. Delete the ith row and column.

Step 3. Define a tournament $T=(t_{ij})_{i\neq j}$ by setting for $i<j$:

$$t_{ij}=i\chi(B_{ij}^{(1)}\geqslant B_{ji}^{(1)})+j\chi(B_{ij}^{(1)}<B_{ji}^{(1)}),\qquad t_{ji}=t_{ij}.$$

Step 4. If T is non-transitive, then STOP. The code word cannot be created.

Step 5. If T is transitive then it has a winner, say k. Replace W by Wk and delete the largest part from each partition in row k.

Step 6. If (B_{ij}) consists only of empty partitions, then STOP. The code word has been found. Otherwise, return to Step 2.

It is important to observe here that because of the way the tournament is defined in Step 3, the matrix $\bar{Q}$ is restricted by the code word W which accompanies it in the following manner: for $1\leqslant i<j\leqslant n$, if the kth letter of W_{ij} is strictly larger than the $(k+1)$st letter, then $\bar{Q}_{ij}^{(k)}$ must be strictly larger than $\bar{Q}_{ij}^{(k+1)}$. Thus, in our example, $W_{23}=2323$ and since the second part of $\bar{Q}_{23}=4210$ came from p_{32} and beat the third part which came from p_{23}, it has to be strictly larger (as indeed it is: $2>1$). The conditions of Bijection M are thus satisfied and we can use it on each partition in $\bar{Q}$. Our pair

$$\bar{Q}=\begin{pmatrix} * & 53321 & 32210\\ & * & 4210\\ & & * \end{pmatrix}\quad W=1123231$$

becomes

$$Q=\begin{pmatrix} * & 42211 & 21100\\ & * & 3110\\ & & * \end{pmatrix}\quad W=1123231.$$

K is the zero matrix which implies that

$$f(K)=(-1)^{\sum_{i<j}k_{ij}}q^{\sum_{i<j}k_{ij}(k_{ij}+1)/2}=1,$$

and so the weight of P is

$$\text{weight}(P)=q ** \Big[\sum_{1\leqslant i\neq j\leqslant n}|p_{ij}|\Big]=q ** \Big[\sum_{1\leqslant i<j\leqslant n}|\bar{Q}_{ij}|\Big]$$

$$= q ** \left[\sum_{1\le i<j\le n} |Q_{ij}| + \sum_{1\le i<j\le n} \mathrm{maj}(W_{ij}) \right]$$

$$= q^{z(W)} \cdot q ** \left[\sum_{1\le i<j\le n} |Q_{ij}| \right].$$

We are now prepared to define a good guy.

Definition. Let $\mathscr{G} = \mathscr{G}(a_1, \ldots, a_n)$ be the set of pairs $(W; Q)$ where W is a word in $M(a_1, \ldots, a_n)$ and $Q = (Q_{ij})_{1\le i<j\le n}$ is an upper triangular matrix of partitions satisfying $\# Q_{ij} = a_i + a_j$. The weight of an element in $\mathscr{G}$ is given by

$$\mathrm{weight}(W; Q) = q^{z(W)} \cdot q ** \left[\sum_{1\le i<j\le n} |Q_{ij}| \right].$$

The elements of $\mathscr{P}$ which do not correspond to good guys will be the bad guys. This includes all elements of $\mathscr{P}$ for which K is not the zero matrix as well as those with K equal to the zero matrix which are interrupted at Step 4 of Algorithm 3.1. The following algorithm transforms our bad guys into a set of objects over which we can sum the weights, and has the desirable feature that if we start with an element of $\mathscr{P}$ which corresponds to a good guy, this algorithm produces the pair $(W; Q)$.

Algorithm 3.2

Step 1. [Initialize] Let W = empty word; $\bar{Q} = (\bar{Q}_{ij})$ = upper triangular matrix of empty partitions; $c_i = 0$ for $1 \le i \le n$; $B = (B_{ij}) = (p_{ij})$; $r_{ij} = a_i + k_{ij}$.

Throughout this algorithm the parameters c_i and r_{ij} will be related to W and B as follows:

$$W \in M(c_1, \ldots, c_n), \qquad \# B_{ij} = r_{ij}.$$

Until we reach Step 5, the number of parts in $\bar{Q}_{ij}$ will be $c_i + c_j$ and r_{ij} will equal $a_i + k_{ij} - c_i$. Since for each i there is always some $j \ne i$ such that $k_{ij} \le 0$, and since $r_{ij} \ge 0$, the last equality implies that c_i is always less than or equal to a_i.

Step 2. [Define tournament] We define a tournament $S = (s_{ij})_{i\ne j}$ only on those values of i for which row i of the matrix B contains at least one non-empty partition. Let $B_{ij}^{(1)}$ be the largest part of the partition B_{ij}; if B_{ij} is empty then define $B_{ij}^{(1)}$ to be $-\infty$. We define S by setting for $i<j$

$$s_{ij} = i\chi(B_{ij}^{(1)} + k_{ij} \ge B_{ji}^{(1)}) + j\chi(B_{ij}^{(1)} + k_{ij} < B_{ji}^{(1)}), \qquad s_{ji} = s_{ij}.$$

If S is non-transitive, go to Step 5. If S is transitive, continue with Step 3.

An important observation at this point is that if $s_{ij} = i$ for any pair $i \ne j$, then B_{ij} is non-empty. To see why this is so, let us assume that $s_{ij} = i$ and B_{ij} is empty. It follows from the definition of s_{ij} that i is less than j and B_{ji} is also empty. Thus,

$$0 = r_{ij} = a_i + k_{ij} - c_i, \qquad 0 = r_{ji} = a_j + k_{ji} - c_j.$$

Summing these equalities and using the fact that $k_{ij} = -k_{ji}$ yields

$$0 = a_i + a_j - (c_i + c_j)$$

which implies that $a_i = c_i$, $a_j = c_j$ since $0 \leqslant c_i \leqslant a_i$, $0 \leqslant c_j \leqslant a_j$.

Since there is a non-empty partition in row i of B, there is an m such that $k_{im} > 0$. Since

$$\sum_j k_{ij} = 0,$$

there is a p such that $k_{ip} < 0$. But then

$$\# B_{ip} = r_{ip} = a_i + k_{ip} - c_i = k_{ip} < 0,$$

a contradiction since no partition can have a strictly negative number of parts.

Step 3. [Find winner and transfer his 'best players' from B to $\bar{Q}$] Since S is transitive, it has a winner i $(1 \leqslant i \leqslant n)$.

(a) For $1 \leqslant m < i$, delete $B_{im}^{(1)}$ from B_{im} and add it as a new part to $\bar{Q}_{mi}$; that is

B_{im} becomes $B_{im}^{(2)} \ldots B_{im}^{(r_{im})}$,
Q_{mi} becomes $\bar{Q}_{mi}^{(1)} \ldots \bar{Q}_{mi}^{(c_m + c_i)} B_{im}^{(1)}$,
and r_{im} is decreased by one.

(b) For $i < m \leqslant n$, delete $B_{im}^{(1)}$ from B_{im} and add a new part $B_{im}^{(1)} + k_{im}$ to $\bar{Q}_{im}$; that is

B_{im} becomes $B_{im}^{(2)} \ldots B_{im}^{(r_{im})}$,
$\bar{Q}_{im}$ becomes $\bar{Q}_{im}^{(1)} \ldots \bar{Q}_{im}^{(c_i + c_m)}(B_{im}^{(1)} + k_{im})$,
and r_{im} is decreased by one.

We now increase c_i by one and replace W by Wi. Note that the conditions at the end of Step 1 are still satisfied.

We observe that if i is the winner $B_{im}^{(1)} + k_{im}$ cannot be strictly negative for if it were then by the definition of s_{im}, B_{mi} is empty which implies that

$$0 = k_{mi} = k_{im}$$

which implies that $B_{im}^{(1)}$ is strictly negative.

Step 4. [Do we have an element of $\mathscr{G}$?] If B contains any non-empty partitions, we return to Step 2. Otherwise, we have combined all the partitions of B into the partitions of $\bar{Q}$ and coded this with W. It remains only to invoke Bijection M on each pair (W_{ij}, Q_{ij}) as described above to yield an element $(W, Q) \in \mathscr{G}$.

Step 5. [Start finalizing an element of $\mathscr{B}$] Let $T = (t_{ij})_{i \neq j}$ be a partial tournament on $\{1, \ldots, n\}$ defined as follows:

(i) if i and j are vertices of S, then $t_{ij} = s_{ij}$,
(ii) if i is a vertex of S and j is not, then $t_{ij} = t_{ji} = i$,
(iii) if neither i nor j is a vertex of S then t_{ij} is undefined.

Recall that $\bar{Q}_{ij} = Q_{ij}^{(1)} \dots Q_{ij}^{(c_i+c_j)}$, $B_{ij} = B_{ij}^{(1)} \dots B_{ij}^{(r_{ij})}$. For each pair (i, j) such that t_{ij} is defined, $i<j$,

(a) If $t_{ij} = i$, then delete $B_{ij}^{(1)}$ from B_{ij} and adjoin a new part $B_{ij}^{(1)} + k_{ij}$ to $\bar{Q}_{ij}$,

$$\bar{Q}_{ij} \text{ becomes } \bar{Q}_{ij}^{(1)} \dots \bar{Q}_{ij}^{(c_i+c_j)}(B_{ij}^{(1)} + k_{ij}),$$

$$B_{ij} \text{ becomes } B_{ij}^{(2)} \dots B_{ij}^{(r_{ij})}.$$

Decrease r_{ij} by one.

(b) If $t_{ij} = j$, then delete $B_{ji}^{(1)}$ from B_{ji} and adjoin a new part $B_{ji}^{(1)}$ to $\bar{Q}_{ij}$, that is

$$\bar{Q}_{ij} \text{ becomes } \bar{Q}_{ij}^{(1)} \dots \bar{Q}_{ij}^{(c_i+c_j)}B_{ji}^{(1)},$$

$$B_{ji} \text{ becomes } B_{ji}^{(2)} \dots B_{ji}^{(r_{ji})}.$$

Decrease r_{ji} by one.

Observe that we now have that,

$$\#\bar{Q}_{ij} = c_i + c_j + \chi(t_{ij} \text{ exists}), \qquad r_{ij} = a_i + k_{ij} - c_i - \chi(t_{ij} = i),$$

where $\chi(t_{ij} = i) = 0$ if t_{ij} does not exist.

Step 6. [Finalize element of $\mathcal{B}$] For each partition $\bar{Q}_{ij}$, the information on whether a given part came from p_{ij} or p_{ji} is encoded by the word $W_{ij}t_{ij}$ where $W_{ij}t_{ij} = W_{ij}$ if t_{ij} does not exist. As in the case of good guys, if the kth letter of $W_{ij}t_{ij}$ is strictly larger than the $(k+1)$st letter, then $\bar{Q}_{ij}^{(k)} > \bar{Q}_{ij}^{(k+1)}$. We apply Bijection M to each pair $(W_{ij}t_{ij}, \bar{Q}_{ij})$ to obtain $(W_{ij}t_{ij}, Q_{ij})$ satisfying

$$|\bar{Q}_{ij}| = |Q_{ij}| + \operatorname{maj}(W_{ij}t_{ij}).$$

We have thus transformed our matrix P into a quadruple $(W, T; Q, B)$ of a word, a non-transitive tournament and two matrices of partitions. We can recover our original matrix P because:

(i) the a_i are known constants,

(ii) the c_i can be recovered from the fact that $\#Q_{ij} = c_i + c_j + \chi(t_{ij} \text{ exists})$, T non-transitive implies that $n \geqslant 3$,

(iii) the k_{ij} can be found using the last relationship of Step 5:

$$k_{ij} = \#B_{ij} - a_i + c_i + \chi(t_{ij} = i),$$

(iv) W and T provide the code for reconstructing $\bar{Q}$ and apportioning the parts in $\bar{Q}$ to recreate P.

A bad guy will be such a quadruple which corresponds to an element $P \in \mathcal{P}$. Specifically, we make the following definition.

Definition. Let $\mathcal{B} = \mathcal{B}(a_1, \dots, a_n)$ be the set of quadruples $(W, T; Q, B)$ such that for some numbers $c_1, \dots, c_n$ with $0 \leqslant c_i \leqslant a_i$ we have

(i) W is word in $M(c_1, \dots, c_n)$;

(ii) $T = (t_{ij})_{i \neq j}$ is a non-transitive partial tournament on $\{1, 2, \dots, n\}$ for which the incomplete vertices (those vertices i for which t_{ij} does not exist for some j)

lose all games and if both i and j are incomplete then t_{ij} does not exist. If W is non-empty then the last letter of W is a spoiler for T. (This follows from the fact that if W is non-empty, then T was formed by taking a transitive tournament and reversing some of the edges to the winner, the last letter of W.)

(iii) $Q=(Q_{ij})_{1\le i<j\le n}$ is an upper triangular matrix of partitions such that $\#Q_{ij}=c_i+c_j+\chi(t_{ij}\text{ exists})$;

(iv) $B=(B_{ij})_{1\le i,j\le n}$ is a matrix of partitions with empty partitions on the diagonal;

(v) Setting $k_{ii}=0$, $k_{ij}=\#B_{ij}-a_i+c_i+\chi(t_{ij}=i)$ for $i\ne j$, we must have that $k_{ij}=-k_{ji}$ and $\sum_j k_{ij}=0$ for each i. Note that this last condition implies that

$$\begin{aligned}0&=\sum_{j\ne i}(\#B_{ij}-a_i+c_i+\chi(t_{ij}=i))\\&=\Big(\sum_j \#B_{ij}\Big)-(n-1)(a_i-c_i)+\sum_j\chi(t_{ij}=i)\end{aligned}$$

and thus the score vector for T is completely determined by B and Q;

(vi) If i is an incomplete vertex of T then $c_i=a_i$ and $k_{ij}=0$ for all j;

(vii) For $1\le i<j\le n$, the smallest part in Q_{ij} is at least as large as the larger of $B_{ij}^{(1)}+k_{ij}+\chi(t_{ij}=i)$ and $B_{ji}^{(1)}$.

For example,

$$W=132\quad T=\begin{pmatrix}t_{12}=1 & t_{13}=3\\ t_{21}=1 & t_{23}=2\\ t_{31}=3 & t_{32}=2\end{pmatrix}$$

$$Q=\begin{pmatrix}* & 444 & 333\\ & * & 555\\ & & *\end{pmatrix}$$

$$B=\begin{pmatrix}* & 33 & 3\\ 4 & * & 4\\ 33 & 5 & *\end{pmatrix}$$

is a member of $\mathscr{B}(3,3,3)$ and we invite the reader to go ahead and check that all the conditions are satisfied and that this corresponds to the matrix

$$P=\begin{pmatrix}* & 4333 & 43\\ 54 & * & 4444\\ 3333 & 65 & *\end{pmatrix}.$$

The weight of a bad guy is the weight of the original P to which it corresponds which is easily checked to be

$$\begin{aligned}&\text{weight}(W,T;Q,B)\\&=f(K)q**\Big[\sum_{i,j}|B_{ij}|+\sum_{i<j}|Q_{ij}|+\sum_{i<j}\text{maj}(W_{ij}t_{ij})-\sum_{i<j}k_{ij}(c_i+\chi(t_{ij}=i))\Big].\end{aligned}$$

We now make T into a complete tournament on $\{1, \ldots, n\}$ by defining $t_{ij} = j$ whenever i and j are incomplete vertices and i is less than j. Note that this does not change the weight of $(W, T; Q, B)$ because if i or j is an incomplete vertex then k_{ij} equals zero.

4. Enumerating the good guys

Theorem 4.

$$\mathrm{weight}(\mathscr{G}) = \frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}} \prod_{1\le i<j\le n} (q)^{-1}_{a_i+a_j}.$$

Proof. Recall that $\mathscr{G} = \mathscr{G}(a_1, \ldots, a_n)$ consists of pairs (W, Q) where W is a member of $M(a_1, \ldots, a_n)$ and $Q = (Q_{ij})_{1\le i<j\le n}$ is an upper triangular partition matrix such that $\#Q_{ij} = a_i + a_j$. The weight is defined by

$$\begin{aligned}\mathrm{weight}(W, Q) &= q ** \left[\sum_{1\le i<j\le n} \mathrm{maj}(W_{ij}) + \sum_{1\le i<j\le n} |Q_{ij}|\right]\\ &= (q ** [z(W)])\left(q ** \left[\sum_{1\le i<j\le n} |Q_{ij}|\right]\right).\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{weight}(\mathscr{G}) &= \sum_{W\in M(a_1,\ldots,a_n)} \sum_{Q} q^{z(W)} \cdot q^{\Sigma|Q_{ij}|}\\ &= \left(\sum_W q^{z(W)}\right)\left(\sum_Q q^{\Sigma|Q_{ij}|}\right) = \left(\sum_{W\in M(a_1,\ldots,a_n)} q^{z(W)}\right) \prod_{1\le i<j\le n} (q)^{-1}_{a_i+a_j},\end{aligned}$$

by the general remarks in Section 1.

Thus Theorem 4 would be proved once the following lemma is proved.

Lemma 4.1.

$$\sum_{W\in M(a_1,\ldots,a_n)} q^{z(W)} = \frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}}$$

(recall that $z(W) = \sum_{1\le i<j\le n} \mathrm{maj}(W_{ij})$*).*

Proof. A venerable principle of mathematics in general and of combinatorics in particular is that of 'structuration' (see Melzak's classic [11, p. 377]), that is, introducing extra structure in order to have more things to hold on to. Every $W \in M(a_1, \ldots, a_n)$ defines a clear-cut permutation $\pi \in S_n$ as follows. Let $\pi(1)$ be the last letter of W. Let $\pi(2)$ be the last letter which is not $\pi(1)$. Let $\pi(3)$ be the last letter that is neither $\pi(1)$ nor $\pi(2)$.... Let $\pi(n)$ be the last letter that is neither $\pi(1)$ ... nor $\pi(n-1)$.

For example if $W = 122144334414$ $\pi(1) = 4$, $\pi(2) = 1$, $\pi(3) = 3$, $\pi(4) = 2$, so

$\pi = 4132$. An alternative definition, which we will find useful later on, is the following: Let $S_{ij} = i$ or j, be the last letter of W_{ij}. This defines a transitive tournament corresponding to a certain permutation π such that $S_{ij} = \pi_{ij}$ (see Section 1 for notation). Thus for the above word

$$\begin{array}{lll} W_{12} = 12211 & W_{13} = 11331 & W_{14} = 1444414 \\ & W_{23} = 2233 & W_{24} = 2244444 \\ & & W_{34} = 33444 \end{array}$$

$$\begin{array}{lll} S_{12} = 1 & S_{13} = 1 & S_{14} = 4 \\ & S_{23} = 3 & S_{24} = 4 \\ & & S_{34} = 4 \end{array}$$

and indeed the winner $\pi(1)$ is 4, $\pi(2) = 1$, $\pi(3) = 3$, $\pi(4) = 2$ so $\pi = 4132$. For every $\pi \in S_n$ let $M_\pi(a_1, \ldots, a_n)$ be the set of words in $M(a_1, \ldots, a_n)$ that yield π. If $a_{\pi(n)}$ is zero then $M_\pi(a_1, \ldots, a_n)$ is equal to the set $M_\pi(a_1, \ldots, \hat{a}_{\pi(n)}, \ldots, a_n)$. If $a_{\pi(n)}$ is not zero but a_i is zero for some $i \neq \pi(n)$, then $M_\pi(a_1, \ldots, a_n)$ is the empty set. Set

$$F(a_1, \ldots, a_n) = \sum_{W \in M(a_1, \ldots, a_n)} q^{z(W)}, \qquad F_\pi(a_1, \ldots, a_n) = \sum_{W \in M_\pi(a_1, \ldots, a_n)} q^{z(W)}.$$

Let us prove

Sublemma 4.1.1. Let

$$c(\pi, a) = c(\pi, a_1, \ldots, a_n) = \sum_{1 \le i < j \le n} a_j \chi(\pi^{-1}(i) < \pi^{-1}(j))$$

and

$$E(\pi) = \prod_{l=2}^{n} [1 - q^{a_{\pi(l)} + \cdots + a_{\pi(n)}}]^{-1},$$

then

$$F_\pi(a_1, \ldots, a_n) = q^{c(\pi, a)} E(\pi) \cdot \frac{(q)_{a_1 + \cdots + a_n - 1}}{(q)_{a_1 - 1} \cdots (q)_{a_n - 1}}.$$

Note that if $a_{\pi(n)}$ is zero then $F_\pi(a_1, \ldots, a_n)$ equals $F_\pi(a_1, \ldots, \hat{a}_{\pi(n)}, \ldots, a_n)$. If $a_{\pi(n)}$ is not zero but a_i is zero for some $i \neq \pi(n)$, then $F_\pi(a_1, \ldots, a_n) = 0$.

Proof of 4.1.1. For any $\pi \in S_n$ and for $l = 1, \ldots, n$ let

$$\begin{aligned} \pi_1 &= \pi = \pi(1)\pi(2) \ldots \pi(n) \\ \pi_2 &= \pi(2)\pi(1)\pi(3) \ldots \pi(n) \\ &\vdots \\ \pi_l &= \pi(2) \ldots \pi(l)\pi(1)\pi(l+1) \ldots \pi(n) \\ &\vdots \\ \pi_n &= \pi(2) \ldots \pi(n)\pi(1). \end{aligned}$$

The sublemma is trivially true for $n=1$. If it is true for $n-1$ it is true for the 'boundary' a's at n, that is, for $(a_1, \ldots, a_n)$ for which $a_i = 0$ for at least one i.

We claim that F_π satisfy the following recurrences:

$$F_\pi(a_1, \ldots, a_{\pi(1)}+1, \ldots, a_n) = \sum_{l=1}^{n} q^{\sum_{r=2}^{l} \chi(\pi(1)<\pi(r))[a_{\pi(1)}+a_{\pi(r)}]} F_{\pi_l}(a_1, \ldots, a_n).$$

This is so because every $W \in M_\pi(a_1, \ldots, a_{\pi(1)}+1, \ldots, a_{\pi(n)})$ ends with the letter $\pi(1)$ and writing $W = W'\pi(1)$ leaves us with $W' \in M_{\pi_l}(a_1, \ldots, a_n)$ for some l between 1 and n.

Since

$$z(W) = z(W'\pi(1)) = z(W') + \sum_{r=2}^{l} (a_{\pi(1)} + a_{\pi(r)})\chi(\pi(1) < \pi(r))$$

the recurrences are explained.

Now we need

Observation 4.1.1.1.

$$c(\pi, a) = \sum_{1 \le i < j \le n} a_{\pi(j)} \chi(\pi(i) < \pi(j)).$$

Proof of 4.1.1.1.

$$\begin{aligned} c(\pi, a) &= \sum_{1 \le i < j \le n} a_j \chi(\pi^{-1}(i) < \pi^{-1}(j)) \\ &= \sum_{i,j} a_{\pi(j)} \chi(i < \pi(j) \text{ and } \pi^{-1}(i) < j) \\ &= \sum_{ij} a_{\pi(j)} \chi(\pi(i) < \pi(j) \text{ and } i < j) \\ &= \sum_{i \le i < j \le n} a_{\pi(j)} \chi(\pi(i) < \pi(j)). \quad \square \end{aligned}$$

Observation 4.1.1.2.

$$\begin{aligned} c(\pi_l; a) = c(\pi; a) &- a_{\pi(2)}\chi(\pi(1) < \pi(2)) \\ &- \cdots - a_{\pi(l)}\chi(\pi(1) < \pi(l)) + a_{\pi(1)}\chi(\pi(2) < \pi(1)) \\ &+ \cdots + a_{\pi(1)}\chi(\pi(l) < \pi(1)). \end{aligned}$$

Proof of 4.1.1.2. Recall that $\pi_l = \pi(2)\pi(3) \ldots \pi(l)\pi(1)\pi(l+1) \ldots \pi(n)$, that is, π_l is obtained from π by moving $\pi(1)$ l steps to the right. In so doing, you lose $\sum_{r=2}^{l} a_{\pi(r)}\chi(\pi(1) < \pi(r))$ but you gain $\sum_{r=2}^{l} a_{\pi(1)}\chi(\pi(r) < \pi(1))$. $\square$

Observation 4.1.1.3. $c(\pi; a_1, \ldots, a_{\pi(1)}+1, \ldots, a_n) = c(\pi; a_1, \ldots, a_n)$.

Proof of 4.1.1.3. Immediate from 4.1.1.1. $\square$

In order to complete the proof of Sublemma 4.1.1 we will show that its r.h.s. satisfies the recurrences established above for the F_π. Namely, we must show that

$$q^{c(\pi;a)}E(\pi)\frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1-1}\cdots(q)_{a_{\pi(1)}}\cdots(q)_{a_n-1}}$$
$$=\sum_{l=1}^{n} q^{H_l}E(\pi_l)\frac{(q)_{a_1+\cdots+a_n-1}}{(q)_{a_1-1}\cdots(q)_{a_n-1}} \qquad (*)$$

where

$$H_l=c(\pi_l;a_1,\ldots,a_n)+\sum_{r=2}^{l}(a_{\pi(1)}+a_{\pi(r)})\chi(\pi(1)<\pi(r)).$$

We need

Observation 4.1.1.4. For $l=1,\ldots,n$

$$H_l=c(\pi;a)+(l-1)a_{\pi(1)}.$$

Proof of 4.1.1.4.

$$H_l=c(\pi_l;a_1,\ldots,a_n)+\sum_{r=2}^{l}(a_{\pi(1)}+a_{\pi(r)})\chi(\pi(1)<\pi(r))$$
$$\overset{(4.1.1.2)}{=} c(\pi;a)-\sum_{r=2}^{l}a_{\pi(r)}\chi(\pi(1)<\pi(r))$$
$$+\sum_{r=2}^{l}a_{\pi(1)}\chi(\pi(r)<\pi(1))+\sum_{r=2}^{l}(a_{\pi(1)}+a_{\pi(r)})\chi(\pi(1)<\pi(r))$$
$$=c(\pi;a)+\sum_{r=2}^{l}a_{\pi(1)}=c(\pi;a)+(l-1)a_{\pi(1)}. \quad \square$$

Dividing both sides of (*) by $q^{c(\pi;a)}(q)_{a_1+\cdots+a_n-1}/((q)_{a_1-1}\cdots(q)_{a_n-1})$ we get that we have to prove

$$\frac{(1-q^{a_1+\cdots+a_n})E(\pi)}{(1-q^{a_{\pi(1)}})}=\sum_{l=1}^{n}q^{(l-1)a_{\pi(1)}}E(\pi_l).$$

Now, for convenience, set $x_l=q^{a_{\pi(l)}}$, $l=1,\ldots,n$ and note that

$$E(\pi_1)=E(\pi)=\frac{1}{(1-x_2\cdots x_n)(1-x_3\cdots x_n)\cdots(1-x_n)},$$

and for $l\geqslant 3$

$$E(\pi_l)=\frac{1}{(1-x_3\cdots x_lx_1x_{l+1}\cdots x_n)\cdots(1-x_n)}.$$

We are left with the task of proving the purely algebraic identity

$$\frac{1-x_1\cdots x_n}{1-x_1}\cdot\frac{1}{(1-x_2\cdots x_n)(1-x_3\cdots x_n)\cdots(1-x_n)}$$

$$=\frac{1}{(1-x_2\cdots x_n)\cdots(1-x_n)}+\frac{x_1}{(1-x_1x_3\cdots x_n)\cdots(1-x_n)}$$
$$+\frac{x_1^2}{(1-x_3x_1x_4\cdots x_n)\cdots(1-x_n)}+\cdots+\frac{x_1^{n-1}}{(1-x_3\cdots x_nx_1)\cdots(1-x_1)}.$$

This will be proved by induction on n:

$$\text{r.h.s.}=\frac{1}{(1-x_2x_3\cdots x_n)\cdots(1-x_n)}+\frac{x_1}{(1-x_1x_3\cdots x_n)}$$
$$\cdot\left[\frac{1}{(1-x_3\cdots x_n)\cdots(1-x_n)}+\cdots+\frac{x_1^{n-2}}{(1-x_4x_5\cdots x_nx_1)\cdots(1-x_1)}\right]$$
$$\overset{\text{inductive}}{\underset{\text{hypothesis}}{=}}\frac{1}{(1-x_2x_3\cdots x_n)\cdots(1-x_n)}$$
$$+\frac{x_1}{(1-x_1x_3\cdots x_n)}\left[\frac{(1-x_1x_3\cdots x_n)}{(1-x_1)}\right]\cdot\frac{1}{(1-x_3\cdots x_n)\cdots(1-x_n)}$$
$$=\left[\frac{1}{(1-x_2\cdots x_n)}+\frac{x_1}{1-x_1}\right]\cdot\frac{1}{(1-x_3\cdots x_n)\cdots(1-x_n)}$$
$$=\frac{(1-x_1\cdots x_n)}{(1-x_1)}\cdot\frac{1}{(1-x_2\cdots x_n)\cdots(1-x_n)}.\quad\square$$

We can now go on to complete the proof of Lemma 4.1. Since $F=\sum_{\pi\in S_n}F_\pi$ we must show that

$$\frac{(q)_{a_1+\cdots+a_n}}{(q)_{a_1}\cdots(q)_{a_n}}=\sum_{\pi\in S_n}q^{c(\pi;a)}\frac{(q)_{a_1+\cdots+a_n-1}}{(q)_{a_1-1}\cdots(q)_{a_n-1}}E(\pi).$$

Dividing by $(q)_{a_1+\cdots+a_n-1}/((q)_{a_1}\cdots(q)_{a_n})$ we must show that

$$(1-q^{a_1+\cdots+a_n})=(1-q^{a_1})\cdots(1-q^{a^n})\sum_{\pi}q^{c(\pi;a)}E(\pi).$$

Letting $x_i=q^{a_i}$, $i=1,\ldots,n$, we are left with proving the purely algebraic identity.

Sublemma 4.1.2. Let

$$d(\pi)=\prod_{1\le i<j\le n}x_{\pi(j)}^{\chi(\pi(i)<\pi(j))}$$

then

$$(1-x_1\cdots x_n)=(1-x_1)\cdots(1-x_n)\sum_{\pi\in S_n}\frac{d(\pi)}{(1-x_{\pi(2)}\cdots x_{\pi(n)})\cdots(1-x_{\pi(n)})}.$$

Proof of 4.1.2. If $\pi(1)=r$, that is, $\pi=r\pi'$ then $d(\pi)=x_{r+1}\cdots x_n d(\pi')$. Now

$$\text{r.h.s.}=(1-x_1)\cdots(1-x_n)\sum_{r=1}^{n}\sum_{\substack{\pi\in S_n\\ \pi(1)=r}}\frac{d(\pi)}{(1-x_{\pi(2)}\cdots x_{\pi(n)})\cdots(1-x_{\pi(n)})}$$

$$=(1-x_1)\cdots(1-x_n)\sum_{r=1}^{n}x_{r+1}\cdots x_n$$

$$\sum_{\pi'\in S_{n-1}}\frac{d(\pi')}{(1-x_{\pi(2)}\cdots x_{\pi(n)})\cdots(1-x_{\pi(n)})}$$

$$=\sum_{r=1}^{n}(1-x_r)x_{r+1}\cdots x_n\sum_{\pi'\in S_{n-1}}\frac{d(\pi')(1-x_1)\cdots\widehat{(1-x_r)}\cdots(1-x_n)}{(1-x_{\pi(2)}\cdots x_{\pi(n)})\cdots(1-x_{\pi(n)})}.$$

Now the inner sum, which ranges over all permutations on $\{1,\ldots,\hat{r},\ldots,n\}$ is by the inductive hypothesis equal to 1, and we are thus left with

$$\sum_{r=1}^{n}(1-x_r)x_{r+1}\cdots x_n \overset{\text{telescoping}}{=} 1-x_1\cdots x_n. \quad \square\square\square$$

5. Getting rid of the bad guys

Theorem 5. $\text{weight}(\mathcal{B})=0$.

Proof. Recall that

$$\text{weight}(W,T;Q,B)=(-1)^{\sum_{1\le i<j\le n}k_{ij}}q^{**}\Bigg[\sum_{1\le i<j\le n}\text{maj}(W_{ij}t_{ij}) + \sum_{1\le i<j\le n}|Q_{ij}|+\sum_{1\le i\ne j\le n}|B_{ij}|+\sum_{1\le i<j\le n}k_{ij}(k_{ij}+1)/2 - \sum_{1\le i<j\le n}k_{ij}(c_i+\chi(t_{ij}=i))\Bigg]$$

where $k_{ij}=c_i+\chi(t_{ij}=i)+r_{ij}-a_i$. From now on let $\text{GAR}(B,Q)$ denote any expression which depends only on B and Q. Since the a_i's are known constants and the c_i's are uniquely determined by Q, anything depending only on the a_i's or c_i's is also included in $\text{GAR}(B,Q)$.

Write $b_{ij}=c_i+r_{ij}-a_i$ then $k_{ij}=b_{ij}+\chi(t_{ij}=i)$ and

$$\sum_{1\le i<j\le n}[(1/2)k_{ij}(k_{ij}+1)-k_{ij}[c_i+\chi(t_{ij}=i)]]$$

$$=\sum_{1\le i<j\le n}[(1/2)(b_{ij}+\chi(t_{ij}=i))(b_{ij}+\chi(t_{ij}=i)+1) -(b_{ij}+\chi(t_{ij}=i))(c_i+\chi(t_{ij}=i))]$$

$$=\text{GAR}-\sum_{1\le i<j\le n}c_i\chi(t_{ij}=i)=\text{GAR}+\sum_{1\le i<j\le n}c_i\chi(t_{ij}=j).$$

Thus

$$\begin{aligned}\text{weight}(W, T; Q, B) &= (-1)^{\text{GAR}+\sum_{1\le i<j\le n}\chi(t_{ij}=i)} \\ &\quad \cdot q ** \left[\text{GAR} + \sum_{1\le i<j\le n} c_i\chi(t_{ij}=j) + \sum_{1\le i<j\le n} \text{maj}(W_{ij}t_{ij})\right] \\ &= \text{GAR}\cdot(-1)^{\sum_{1\le i<j\le n}\chi(t_{ij}=i)} \\ &\quad \cdot q ** \left[\sum_{1\le i<j\le n} c_j\chi(t_{ij}=i) + \sum_{1\le i<j\le n} \text{maj}(W_{ij}t_{ij})\right].\end{aligned}$$

Now

$$\begin{aligned}\text{weight}(\mathscr{B}) &= \sum_{Q,B}\sum_{W,T} \text{weight}(W, T; Q, B) \\ &= \sum_{Q,B} (-1)^{\text{GAR}} q^{\text{GAR}} \sum_{W,T} (-1)^{\sum_{1\le i<j\le n}\chi(t_{ij}=j)} \\ &\quad \cdot q ** \left[\sum_{1\le i<j\le n} c_i\chi(t_{ij}=j) + \sum_{1\le i<j\le n} \text{maj}(W_{ij}t_{ij})\right].\end{aligned}$$

The theorem would be proven if we can show that the inner sum is always zero, that is, for every n; $c_1, \ldots, c_n$ and score vector $\bar{w}$ we must show

$$\sum_{W,T} (-1)^{\sum_{1\le i<j\le n}\chi(t_{ij}=j)} \cdot q ** \left[\sum_{1\le i<j\le n} c_i\chi(t_{ij}=j) + \sum_{1\le i<j\le n} \text{maj}(W_{ij}t_{ij})\right] = 0,$$

where $\sum_{W,T}$ is the sum over all $W \in M(c_1, \ldots, c_n)$ and non-transitive T with score vector $\bar{w}$ such that the last letter of W is a spoiler for T. If W is empty, then our sum is merely

$$\sum_T (-1)^{\sum_{i<j}\chi(t_{ij}=j)}$$

where T does not have to have a spoiler. This sum is zero by the involution on non-transitive tournaments given in Gessel's paper [6]. We shall therefore assume that W is non-empty. We call the term in the sum on W and T term(W, T) and we see that

$$\sum_{W,T} \text{term}(W, T) = \sum_{\pi\in S_n} \sum_{\substack{W\in M_\pi(c_1,\ldots,c_n) \\ T\in \text{NonTrans}(n;\bar{w};\pi(1))}} \text{term}(W, T)$$

where NonTrans$(n; \bar{w}; \pi(1))$ is the set of non-transitive tournaments on $\{1, \ldots, n\}$ with score vector $\bar{w}$ and spoiler $\pi(1)$.

But if $W \in M_\pi(c_1, \ldots, c_n)$

$$\sum_{1\le i<j\le n} \text{maj}(W_{ij}t_{ij}) = \sum_{1\le i<j\le n} \text{maj}(W_{ij}) + \sum_{1\le i<j\le n} (c_i + c_j)\chi(\pi_{ij} > t_{ij}),$$

where π_{ij} is the transitive tournament corresponding to π:

$$\pi_{ij} = i \iff \pi^{-1}(i) < \pi^{-1}(j).$$

Thus the above sum can be written

$$\sum_{\pi \in S_n} \sum_{T \in \mathrm{NonTrans}(n)} (-1)^{\sum_{1 \leqslant i<j \leqslant n} \chi(t_{ij}=i)}$$

$$\cdot q ** \left[\sum_{1 \leqslant i<j \leqslant n} c_i \chi(t_{ij}=j) + \sum_{1 \leqslant i<j \leqslant n} (c_i+c_j)\chi(\pi_{ij}>t_{ij}) \right]$$

$$\sum_{W \in M_\pi(c_1,\ldots,c_n)} q^{\sum_{1 \leqslant i<j \leqslant n} \mathrm{maj}(W_{ij})}.$$

Now by Sublemma 4.1.1

$$\sum_{W \in M_\pi(c_1,\ldots,c_n)} q^{\sum_{1 \leqslant i<j \leqslant n} \mathrm{maj}(W_{ij})} = F_\pi(c_1,\ldots,c_n)$$

$$= \frac{(q)_{c_1+\cdots+c_n-1}}{(q)_{c_1-1} \cdots (q)_{c_n-1}} \cdot \frac{q^{\sum_{1 \leqslant i<j \leqslant n} c_j \chi(\pi_{ij}=i)}}{(1-q^{c_{\pi(2)}+\cdots+c_{\pi(n)}}) \cdots (1-q^{c_{\pi(n)}})}.$$

Substituting above we get that the sum is equal to

$$\frac{(q)_{c_1+\cdots+c_n-1}}{(q)_{c_1-1} \cdots (q)_{c_n-1}} \sum_{\substack{\pi \in S_n \\ T \in \mathrm{NonTrans}(n)}} (-1)^{\sum_{1 \leqslant i<j \leqslant n} \chi(t_{ij}=i)}$$

$$\cdot q ** \left[\sum_{1 \leqslant i<j \leqslant n} c_j \chi(\pi_{ij}=i) + c_i \chi(t_{ij}=j) + (c_i+c_j)\chi(\pi_{ij}>t_{ij}) \right] \cdot E(\pi)$$

where $E(\pi) = (1-q^{c_{\pi(2)}+\cdots+c_{\pi(n)}})^{-1} \cdots (1-q^{c_{\pi(n)}})^{-1}$.

Introducing the notation $y_i = q^{c_i}$ $(i=1,\ldots,n)$ we are left with the task of proving the following purely algebraic identity.

Lemma 5.1. *Let, for $\pi \in S_n$ and $T \in \mathrm{NonTrans}(n; \bar{w}; \pi(1))$*

$$\mathrm{weight}(\pi, T) = \frac{(-1)^{\sum_{1 \leqslant i<j \leqslant n} \chi(t_{ij}=j)} \prod_{1 \leqslant i<j \leqslant n} y_j^{\chi(\pi_{ij}=i)} y_i^{\chi(t_{ij}=j)} (y_i y_j)^{\chi(\pi_{ij}>t_{ij})}}{(1-y_{\pi(2)} \cdots y_{\pi(n)})(1-y_{\pi(3)} \cdots y_{\pi(n)}) \cdots (1-y_{\pi(n)})}$$

then

$$\sum_{\substack{\pi \in S_n \\ T \in \mathrm{NonTrans}(n;\bar{w};\pi(1))}} \mathrm{weight}(\pi, T) = 0.$$

Proof. We shall use the fact that T is almost transitive, Write $\pi = r\pi'$ where $r = \pi(1)$ and π' is a permutation on $\{1,\ldots,r-1,r+1,\ldots,n\}$ and let σ be the transitive tournament obtained from T by deleting r. The transitive tournament σ will also be identified with the permutation it defines. Let L_rT denote the number of players who beat r in tournament T, $L_rT = \sum_{j=1, j \neq r}^{n} \chi(t_{rj}=j)$. We make the following observation:

Observation 5.1.1.

$$|\mathrm{weight}(\pi, T)| = \frac{|\mathrm{weight}(\pi', \sigma)| \cdot (y_1 y_2 \cdots \hat{y}_r \cdots y_n) y_r^{L_rT}}{(1-y_{\pi(2)} \cdots y_{\pi(n)})}.$$

Proof of 5.1.1.

$$|\mathrm{weight}(\pi, T)| = \frac{\prod_{1\le i<j\le n} y_j^{\chi(\pi_{ij}=i)} y_i^{\chi(t_{ij}=j)} (y_i y_j)^{\chi(\pi_{ij}>t_{ij})}}{(1-y_{\pi(2)}\cdots y_{\pi(n)})\cdots(1-y_{\pi(n)})}$$

$$= \frac{|\mathrm{weight}(\pi', \sigma)|}{(1-y_{\pi(2)}\cdots y_{\pi(n)})} \prod_{j=r+1}^{n} y_j^{\chi(\pi_{rj}=r)} \prod_{i=1}^{r-1} (y_i y_r)^{\chi(t_{ir}=i)} \prod_{i=1}^{r-1} y_i^{\chi(t_{ir}=r)} \prod_{j=r+1}^{n} y_r^{\chi(t_{rj}=j)}$$

$$= \frac{|\mathrm{weight}(\pi', \sigma)|\, y_{r+1}\cdots y_n}{(1-y_{\pi(2)}\cdots y_{\pi(n)})} \prod_{i=1}^{r-1} (y_i y_r)^{\chi(t_{ir}=i)} \prod_{i=1}^{r-1} y_i^{\chi(t_{ir}=r)} \prod_{j=r+1}^{n} y_r^{\chi(t_{rj}=j)}$$

$$= \frac{|\mathrm{weight}(\pi', \sigma)|\, y_{r+1}\cdots y_n}{(1-y_{\pi(2)}\cdots y_{\pi(n)})} \prod_{i=1}^{r-1} y_i^{\chi(t_{ir}=i)+\chi(t_{ir}=r)} \prod_{i=1}^{r-1} y_r^{\chi(t_{ir}=i)} \prod_{j=r+1}^{n} y_r^{\chi(t_{rj}=j)}$$

$$= \frac{|\mathrm{weight}(\pi', \sigma)|}{(1-y_{\pi(2)}\cdots y_{\pi(n)})} (y_1\cdots \hat{y}_r \cdots y_n) y_r^{L_r T}. \quad \square$$

We shall also need the following observation. Here we take σ to be a transitive tournament.

Observation 5.1.2. For every $\sigma \in S_n$

$$\sum_{\pi\in S_n} \mathrm{weight}(\pi, \sigma) = \frac{\mathrm{sgn}(\sigma)(1-y_1\cdots y_n) y_{\sigma(1)}^0 y_{\sigma(2)}^1 \cdots y_{\sigma(n)}^{n-1}}{(1-y_1)\cdots(1-y_n)}.$$

Proof of 5.1.2. This is trivially true for $n=1$. We shall proceed by induction using 5.1.1. with σ' being the permutation obtained from σ by deleting $r=\pi(1)$. Note that since σ is a permutation, $L_r\sigma$ equals $\sigma^{-1}(r)-1$. Now

$$\mathrm{sgn}(\sigma) \sum_{\pi\in S_n} \mathrm{weight}(\pi, \sigma) = \sum_{\pi\in S_n} |\mathrm{weight}(\pi, \sigma)| = \sum_{r=1}^{n} \sum_{\substack{\pi\in S_n\\ \pi(1)=r}} |\mathrm{weight}(\pi, \sigma)|$$

$$\overset{(5.1.1)}{=} \sum_{r=1}^{n} \frac{(y_1\cdots\hat{y}_r\cdots y_n) y_r^{\sigma^{-1}(r)-1}}{(1-y_{\pi(2)}\cdots y_{\pi(n)})} \sum_{\pi'\in S_{n-1}(r)} |\mathrm{weight}(\pi', \sigma')|$$

$$\overset{\text{induction}}{=} \sum_{r=1}^{n} (y_1\cdots\hat{y}_r\cdots y_n) y_r^{\sigma^{-1}(r)-1} \frac{y_{\sigma'(1)}^0 \cdots y_{\sigma'(n-1)}^{n-2}}{(1-y_1)\cdots\widehat{(1-y_r)}\cdots(1-y_n)}$$

$$= \frac{1}{(1-y_1)\cdots(1-y_n)} \sum_{r=1}^{n} y_{\sigma(1)}\cdots y_{\sigma(\sigma^{-1}(r)-1)}(1-y_r)\cdot y_{\sigma(1)}^0\cdots y_{\sigma(n)}^{n-1}$$

$$= \frac{y_{\sigma(1)}^0 y_{\sigma(2)}^1\cdots y_{\sigma(n)}^{n-1}}{(1-y_1)\cdots(1-y_n)} \sum_{r=1}^{n} y_{\sigma(1)}\cdots y_{\sigma(\sigma^{-1}(r)-1)}(1-y_{\sigma(\sigma^{-1}(r))})$$

$$\overset{\text{telescoping}}{=} \frac{y_{\sigma(1)}^0\cdots y_{\sigma(n)}^{n-1}}{(1-y_1)\cdots(1-y_n)} (1-y_{\sigma(1)}\cdots y_{\sigma(n)})$$

$$\overset{\text{commutativity}}{=} \frac{y_{\sigma(1)}^0\cdots y_{\sigma(n)}^{n-1}}{(1-y_1)\cdots(1-y_n)} (1-y_1\cdots y_n). \quad \square$$

We now combine the observations in order to simplify the sum which is to be shown to be zero. Recall that σ is T with its spoiler removed.

$$\sum_{\substack{\pi\in S_n \\ T\in\mathrm{NonTrans}(n;\bar{w};\pi(1))}} \mathrm{weight}(\pi, T) = \sum_{r=1}^{n} \sum_{T\in\mathrm{NonTrans}(n;\bar{w};r)} \sum_{\substack{\pi\in S_n \\ \pi(1)=r}} \mathrm{weight}(\pi, T)$$

$$\overset{(5.1.1)}{=} \sum_{r=1}^{n} \sum_{T\in\mathrm{NonTrans}(n;\bar{w};r)} \sum_{\pi'\in S_{n-1}(\hat{r})} (-1)^{\sum_{1\le i<j\le n}\chi(t_{ij}=j)} \cdot \frac{(|\mathrm{weight}(\pi', \sigma)|\,(y_1\cdots\hat{y}_r\cdots y_n)y_r^{L_rT}}{(1-y_{\pi(2)}\cdots y_{\pi(n)})}$$

$$= \sum_{r=1}^{n} \sum_{T\in\mathrm{NonTrans}(n;\bar{w};r)} (-1)^{\sum_{1\le i<j\le n}\chi(t_{ij}=j)} \cdot \frac{(y_1\cdots\hat{y}_r\cdots y_n)y_r^{L_rT}}{(1-y_1\cdots\hat{y}_r\cdots y_n)} \sum_{\pi'\in S_{n-1}(\hat{r})} \mathrm{sgn}(\sigma)\mathrm{weight}(\pi', \sigma)$$

$$\overset{(5.1.2)}{=} \sum_{r=1}^{n} \sum_{T\in\mathrm{NonTrans}(n;\bar{w};r)} (-1)^{\sum_{1\le i<j\le n}\chi(t_{ij}=j)} \cdot \frac{(y_1\cdots\hat{y}_r\cdots y_n)y_r^{L_rT}y_{\sigma(1)}^{0}y_{\sigma(2)}^{1}\cdots y_{\sigma(n-1)}^{n-2}}{(1-y_1)\cdots\widehat{(1-y_r)}\cdots(1-y_n)}$$

$$= \prod_{i=1}^{n}\frac{1}{(1-y_i)} \sum_{r=1}^{n} \sum_{T\in\mathrm{NonTrans}(n;\bar{w};r)} (-1)^{\sum_{1\le i<j\le n}\chi(t_{ij}=j)} y_r^{L_rT}(1-y_r)y_{\sigma(1)}^{1}y_{\sigma(2)}^{2}\cdots y_{\sigma(n-1)}^{n-1}.$$

It now only remains to be shown that

$$\sum_{r=1}^{n} \sum_{T\in\mathrm{NonTrans}(n;\bar{w};r)} (-1)^{\sum_{1\le i<j\le n}\chi(t_{ij}=j)} y_r^{L_rT}y_{\sigma(1)}^{1}\cdots y_{\sigma(n-1)}^{n-1}$$
$$-\sum_{r=1}^{n} \sum_{T\in\mathrm{NonTrans}(n;\bar{w};r)} (-1)^{\sum_{1\le i<j\le n}\chi(t_{ij}=j)} y_r^{L_rT+1}y_{\sigma(1)}^{1}\cdots y_{\sigma(n-1)}^{n-1} = 0.$$

We shall prove this by exhibiting an involution on all elements over which we are summing which preserves the absolute value of the weight and reverses the sign.

We observe that since there is at least one cycle through r, we have bounds on $L_rT: 1 \le L_rT \le n-2$. We first define the involution on the set of pairs (r, T), $T\in\mathrm{NonTrans}(n;\bar{w};r)$, which have weight

$$(-1)^{\sum_{1\le i<j\le n}\chi(t_{ij}=j)} y_r^{L_rT}y_{\sigma(1)}^{1}\cdots y_{\sigma(n-1)}^{n-1}$$

and for which r beats $\sigma(L_rT)$, Player $\sigma(L_rT)$ loses precisely L_rT matches since he loses to $\sigma(1), \sigma(2), \ldots, \sigma(L_rT-1)$ and to r. Therefore if we simply exchange the labels of players r and $\sigma(L_rT)$, so that $\sigma(L_rT)$ now becomes the spoiler, we have not changed the score vector or the absolute value of the weight. But since

$\sigma(L_rT)$ now beats r we have changed the sign.

$$\begin{array}{ll}
\sigma(1) & \sigma(1) \\
\downarrow & \downarrow \\
\sigma(2) & \sigma(2) \\
\downarrow & \downarrow \\
\vdots & \vdots \\
\downarrow & \downarrow \\
\sigma(L_rT) \leftarrow r & r \leftarrow \sigma(L_rT) \\
\downarrow & \downarrow \\
\vdots & \vdots
\end{array}$$

The second involution is on the set of pairs (r, T), $T \in \mathrm{NonTrans}(n; \bar{w}; r)$, which have weight

$$-(-1)^{\sum_{1\le i<j\le n}\chi(t_{ij}=j)} y_r^{L_rT+1} y_{\sigma(1)}^1 \cdots y_{\sigma(n-1)}^{n-1}$$

and for which $\sigma(L_rT+1)$ beats r. Player $\sigma(L_rT+1)$ loses precisely L_rT matches since he loses to $\sigma(1), \ldots, \sigma(L_rT)$ but not to r. Therefore if we exchange the labels of players r and $\sigma(L_rT+1)$, so that $\sigma(L_rT+1)$ now becomes the spoiler, we have not changed the score vector or the absolute value of the weight, but we have changed the sign.

$$\begin{array}{ll}
\sigma(1) & \sigma(1) \\
\downarrow & \downarrow \\
\vdots & \vdots \\
\downarrow & \downarrow \\
\sigma(L_rT+1) \rightarrow r & r \rightarrow \sigma(L_rT+1) \\
\downarrow & \downarrow \\
\vdots & \vdots
\end{array}$$

We now define a sign reversing bijection between the two remaining sets. Let (r, T), $T \in \mathrm{NonTrans}(n; \bar{w}; r)$, have weight

$$(-1)^{\sum_{1\le i<j\le n}\chi(t_{ij}=j)} y_r^{L_rT} y_{\sigma(1)}^1 \cdots y_{\sigma(n-1)}^{n-1}$$

and be such that $\sigma(L_rT)$ beats r. Note that $L_rT \neq 1$ for if it were then $\sigma(1)$ beats r and r beats everyone else and so the tournament is transitive. For the same reason, there must be a $j > L_rT$ such that $\sigma(j)$ beats r. We exchange the labels of players r and $\sigma(L_rT)$ and reverse the arrow between these two players, so $\sigma(L_rT)$ still beats r. If we let $s = \sigma(L_rT)$ be the new spoiler, U be the new tournament and τ the transitive tournament obtained from U by deleting s, then $L_sU = L_rT - 1$. Our new pair (s, U) has the same score vector as (r, T) and now $s = \sigma(L_rT)$ beats $\tau(L_sU+1) = \tau(L_rT) = r$. The weight of (s, U) is

$$(-1)^{\sum_{1\le i<j\le n}\chi(t_{ij}=j)} y_s^{L_sU+1} y_{\tau(1)}^1 \cdots y_{\tau(n-1)}^{n-1},$$

precisely the negative of the weight assigned to such a pair in the second summation. Any pair (s, U) of the second sum for which s beats $\tau(L_sU+1)$ must arise in this manner since $L_s U$ cannot equal $n-2$, for if it did then s would beat $\tau(n-1)$ and lose to everyone else and so U would be transitive. Also, since U is non-transitive, there must be a $j \leqslant L_s U$ such that s beats $\tau(j)$.

$$
\begin{array}{lll}
\sigma(1) & \qquad\qquad & \sigma(1) = \tau(1) \\
\downarrow & & \quad\downarrow \\
\vdots & & \quad\vdots \\
\downarrow & & \quad\downarrow \\
\sigma(L_r T) \rightarrow r & & r = \tau(L_s U + 1) \leftarrow \sigma(L_r T) = s \\
\downarrow & & \quad\downarrow \\
\vdots & & \quad\vdots
\end{array}
\qquad \square\square
$$

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