NOTE

A COMBINATORIAL PROOF OF
NEWTON'S IDENTITIES

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We are going to give a new proof of Newton's celebrated identities

\[ \sum_{r=0}^{k-1} (-1)^r \left( \sum_{1 \leq i_1 < \ldots < i_r \leq n} x_{i_1} \cdots x_{i_r} \right) \left( \sum_{j=1}^n x_j^{k-r} \right) + (-1)^k \left( \sum_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1} \cdots x_{i_k} \right) = 0, \quad (*) \]

where \( n \) and \( k \) are positive integers and \( x_1, \ldots, x_n \) are commuting indeterminates.

Consider the set \( \mathcal{A} = \mathcal{A}(n, k) \) of pairs \((A, j)\) where

(i) \( A \) is a subset of \( \{1, \ldots, n\} \),

(ii) \( j \) is a member of \( \{1, \ldots, n\} \),

(iii) \( |A| + l = k \), where \( |A| \) denotes the number of elements of \( A \),

(iv) \( l \geq 0 \) and if \( l = 0 \), then \( j \in A \).

Define the weight of \((A, j)\), \( w(A, j) \) by \( w(A, j) = (-1)^{|A|} (\prod_{a \in A} x_a) x_j^{|j|} \), for example \( w(\{1, 3, 5\}, 2^3) = (-1)^3 x_1 x_3 x_5 \cdot x_2^3 = -x_1 x_3^2 x_5 \). It is readily seen that the l.h.s. of \((*)\) is the sum of all the weights of the elements of \( \mathcal{A} \). We will now prove that this sum is zero. To this end introduce the mapping \( T : \mathcal{A} \rightarrow \mathcal{A} \) defined by

\[ T(A, j) = \begin{cases} (A/\{j\}, j^{l+1}) & j \in A, \\ (A \cup \{j\}, j^{l-1}) & j \notin A. \end{cases} \]

This mapping satisfies \( w(T(A, j)) = -w(A, j) \) and is an involution (i.e. \( T^2 = \text{identity} \)). Thus all the weights can be arranged in mutually cancelling pairs and their sum is therefore zero.