SIAM J. MATH. ANAL.
Vol. 14, No. 2, March 1983

0036-1410/83/1402-0014 $01.25/0

ON ELEMENTARY METHODS IN POSITIVITY THEORY*

J. GILLIS,† B. REZNICK‡ AND D. ZEILBERGER§

Abstract. We give a short proof of a result of Askey and Gasper [J. Analyse Math., 31 (1977), pp. 48–68] that $(1-x-y-z+4xyz)^{-\beta}$ has positive power series coefficients for $\beta\geq(\sqrt{17}-3)/2$. We also show how Ismail and Tamhankar's proof [SIAM J. Math. Anal., 10 (1979), pp. 478–485] that

$$(1-(1-\lambda)x-\lambda y-\lambda xz-(1-\lambda)yz+xyz)^{-\alpha}\qquad(0\leq\lambda\leq 1)$$

has positive power series coefficients for $\alpha=1$ implies Koornwinder's result that it does so for $\alpha\geq 1$.

1. Introduction. Given a multivariate polynomial $P(x_1,\cdots,x_n)$ and a real β, it is of interest to know whether $P^{-\beta}$ has only positive terms in its power series expansion. Szegö [6] proved that this was the case for $P=(1-x)(1-y)+(1-x)(1-z)+(1-y)\cdot(1-z)$ and $\beta\geq\frac{1}{2}$, and Askey and Gasper [2] established positivity for $P=1-x-y-z+4xyz$ and $\beta\geq(\sqrt{17}-3)/2$. A fascinating account of the history of these problems up to 1975 is given in Askey's monograph [1].

Koornwinder [5] used deep methods to establish the positivity of the coefficients of $[1-(1-\lambda)x-\lambda y-\lambda xz-(1-\lambda)yz+xyz]^{-\beta}$ for $0\leq\lambda\leq 1$, $\beta\geq 1$ and that of $[1-x-y-z-u+4(xyz+xyu+xzu+yzu)-16xyzu]^{-\beta}$. Later, Ismail and Tamhankar [4] (see also [3]) gave elementary proofs of Koornwinder's results in the special case $\beta=1$. In §2 we are going to show how Ismail and Tamhankar's results for $\beta=1$ imply Koornwinder's results for $\beta\geq 1$ and in §3 we give a short proof of Askey and Gasper's [2] result. Finally, in §4 we conjecture that for $n\geq 4$, $(1-(x_1+\cdots+x_n)+n!x_1\cdots x_n)^{-1}$ has positive coefficients.

2. Operations that preserve positivity of coefficients.

PROPOSITION 1. *Suppose that* $a(x_1,\cdots,x_{n-1})$ *and* $b(x_1,\cdots,x_{n-1})$ *are polynomials. If* (i) $(a-bx_n)^{-1}$ *has positive coefficients and* (ii) $a^{-\alpha}$ *has positive coefficients for all* $\alpha>0$, *then so does* $(a-bx_n)^{-\beta}$ *for all* $\beta\geq 1$.

Proof. By hypothesis $(a-bx_n)^{-1}=\Sigma(b^r/a^{r+1})x_n^r$ has positive coefficients, implying that for every r, b^r/a^{r+1} has positive coefficients. Since $(\beta)_r/r!=\beta(\beta+1)\cdots(\beta+r-1)/r!$ is positive and $a^{1-\beta}$ has positive coefficients, we see that

$$(a-bx_n)^{-\beta}=a^{1-\beta}\sum_{r=0}^{\infty}\frac{(\beta)_r}{r!}\frac{b^r}{a^{r+1}}$$

has positive coefficients. □

By taking $a=1-(1-\lambda)x-\lambda y$, $b=\lambda x+(1-\lambda)y-xy$ $(0\leq\lambda\leq 1)$ it follows that Ismail and Tamhankar's result that $[1-(1-\lambda)x-\lambda y-\lambda xz-(1-\lambda)yz+xyz]^{-\beta}$ $(0\leq\lambda\leq 1)$ has positive coefficients for $\beta=1$ implies Koornwinder's result that it does so for $\beta\geq 1$.

PROPOSITION 2. *If* $[a(x,y)-b(x,y)z]^{-\alpha}$ *and* $[c(x,y)-d(x,y)z]^{-\alpha}$ *have positive coefficients* $(\alpha>0)$ *so also does* $[a(x,y)c(z,u)-b(x,y)d(z,u)]^{-\alpha}$.

* Received by the editors June 15, 1981, and in final revised form January 26, 1982. This research was supported in part by the National Science Foundation.
† Department of Applied Mathematics, The Weizmann Institute of Science, Rehovot, Israel.
‡ Department of Mathematics, University of Illinois, Urbana, Illinois 61801.
§ Department of Theoretical Mathematics, The Weizmann Institute of Science, Rehovot, Israel.

Proof. $(a-bz)^{-\alpha}=a^{1-\alpha}\Sigma[(\alpha)_r/r!](b^r/a^{r+1})z^r$ and $(c-dz)^{-\alpha}=c^{1-\alpha}\Sigma[(\alpha)_r/r!](d^r/c^{r+1})z^r$ have positive coefficients. Thus for every r, both $a^{1-\alpha}b^r/a^{r+1}$ and $c^{1-\alpha}d^r/c^{r+1}$ do and, hence, does $(ac)^{1-\alpha}b^rd^r/a^{r+1}c^{r+1}$ and finally does

$$(ac)^{1-\alpha}\sum\frac{(\alpha)_r}{r!}\frac{b^rd^r}{a^{r+1}c^{r+1}}=(a(x,y)c(z,u)-b(x,y)d(z,u))^{-\alpha}.$$

Take $a(x,y)=1-x-y$, $b(x,y)=x+y-4xy$, $c(z,u)=1-z-ud$, $d(z,u)=z+u-4zu$. The hypotheses of Proposition 2 are satisfied (for $\alpha\geq 1$) by virtue of the above discussion, (with $\lambda=\frac{1}{2}$, $x\leftarrow 2x$, $y\leftarrow 2y$). Thus, we have an elementary proof of Koornwinder's [5] result that $[1-x-y-z-u+4(xyu+xyz+xzu+yzu)-16xyzu]^{-\alpha}$ has positive coefficients for $\alpha\geq 1$.

3. A short proof of a result of Askey and Gasper. It follows from the above that $(1-x-y-z+4xyz)^{-\beta}$ has positive coefficients for $\beta\geq 1$. Askey and Gasper [2] extended this result to $\beta\geq(\sqrt{17}-3)/2$. This can be obtained quite simply by an extension of a method used in [3].

Suppose that $\beta>(\sqrt{17}-3)/2$. Write $R=1-x-y-z+4xyz$, it is readily seen that

$$\frac{\partial}{\partial x}R^{-\beta}=(1+2z)\left[x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\beta\right]R^{-\beta}+2\left(y\frac{\partial}{\partial y}-z\frac{\partial}{\partial z}\right)R^{-\beta}.$$

Substitute $R^{-\beta}=\Sigma D_{a+1,b,c}x^{a+1}y^bz^c$ above, compare coefficients of $x^ay^bz^c$, and set $a\leftarrow a-1$ to get

$$aD_{a,b,c}=(a+b-c+\beta-1)D_{a-1,b,c}+2(a-b+c-2+\beta)D_{a-1,b,c-1}.$$

Now, by symmetry, it is enough to prove positivity for $a\geq b\geq c$. The coefficients of the above recurrence are positive if $a\geq b\geq c>1$ and the result will follow by induction if $D_{a,a,1}\geq 0$ for all a. Now

$$D_{a,a,1}=\frac{\beta(\beta+1)\cdots(\beta+2a-2)}{(a-1)!^2}\left[\frac{(\beta+2a-1)(\beta+2a)}{a^2}-4\right].$$

But $(\beta+2a-1)(\beta+2a)-4a^2=\beta^2-\beta+2a(2\beta-1)$ increases with a since $\beta\cong 0.56>0.5$ and $D_{1,1,1}=\beta(\beta^2+3\beta-2)>0$, so the result follows.

4. Does $(1-(x_1+\cdots+x_n)+n!x_1\cdots x_n)^{-1}$ have positive power series coefficients? We have already mentioned Askey and Gasper's result that $[1-(x+y+z)+4xyz]^{-1}$ has positive power series coefficients. We are interested in A_n, the largest A for which $(1-(x_1+\cdots+x_n)+Ax_1\cdots x_n)^{-1}$ has nonnegative coefficients. Since the coefficients of $x_1\cdots x_n$ in the above expansion is $n!-A_n$, we must certainly have $A_n\leq n!$. We conjecture that for $n\geq 4$, $A_n=n!$. It may be seen that $A_n\geq(n-1)!$, i.e., that $[1-(x_1+\cdots+x_n)+(n-1)!x_1\cdots x_n]^{-1}$ has positive coefficients. The reason is that the coefficient of $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ in the above expansion has combinatorial significance, namely, it is the number of words with α_1 1's,$\cdots$,α_n n's such that no substring of n letters which ends with the letter "n" can be a permutation (e.g., with $n=4$, the six words 1234, 1324, 2134, 2314, 3124, 3214 are not allowed as subwords) (see Zeilberger [7] for details).

Let us state:

PROPOSITION 3. *Let* $(1-(x_1+\cdots+x_n)+n!x_1\cdots x_n)^{-1}=\Sigma A_{\alpha_1,\cdots,\alpha_n}x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. *If* $A_{r,\cdots,r}\geq 0$ *for all* r, *then* $A_{\alpha_1\cdots\alpha_n}\geq 0$ *for all* $(\alpha_1,\cdots,\alpha_n)\in N^n$.

The proof is rather long and we omit it here. Note that

$$A^{(n)}_{r,\dots,r} = \sum_{j=0}^{r} (-1)^j \frac{(rn-(n-1)j)!(n!)^j}{(r-j)!^n j!},$$

and it would therefore suffice to show that this binomial sum is positive. This has been verified by computer for $n=4$ and $1 \le r \le 220$. In this range $A^{(4)}_{r,\dots,r}$ increases monotonically and appears to have exponential growth. This supports our conjecture.

Acknowledgment. Many thanks are due to Gilad Bandel for his programming.

REFERENCES

[1] R. Askey, *Orthogonal Polynomials and Special Functions*, CBMS Regional Conference Series in Applied Mathematics 21, Society for Industrial and Applied Mathematics, Philadelphia, 1975.
[2] R. Askey and G. Gasper, *Convolution structures of Laguerre polynomials*, J. Analyse Math., 31 (1977), pp. 48–68.
[3] J. Gillis and J. Kleeman, *A combinatorial proof of a positivity result*, Math. Proc. Camb. Phil. Soc., 86 (1979), pp. 13–19.
[4] M. E. H. Ismail and M. V. Tamhankar, *A combinatorial approach to some positivity problems*, SIAM J. Math. Anal., 10 (1979), pp. 478–485.
[5] T. Koornwinder, *Positivity proofs for linearization and connection coefficients of orthogonal polynomials satisfying an addition theorem*, J. London Math. Soc., 2 (1978), pp. 101–114.
[6] G. Szego, *Über gewisse Potenzeihen mit lauter positiven Koeffizienten*, Math. Z., 37 (1933), pp. 674–688.
[7] D. Zeilberger, *Enumeration of words by their number of mistakes*, Discrete Math., 34 (1981), pp. 89–91.