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Mathematics

Pompeiu's problem on discrete space

(partial difference equations/Hilbert's *Nullstellensatz*/three squares theorem)

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ABSTRACT Let $\mathcal{S}$ be a finite family of finite subsets of the n-dimensional lattice Z^n, and let $\mathcal{T}$ denote the group of all translations of Z^n. We shall here consider the *Pompeiu problem* for the family $\mathcal{S}$—namely, to determine when the only function $f{:}Z^n \to C$ such that

$$\sum_{m \in \tau(S)} f(m) = 0 \text{ for all } \tau \in \mathcal{T} \text{ and } S \in \mathcal{S}$$

is the zero function.

1. Introduction and preliminaries

Let $\mathcal{S}$ be a family of compact subsets of the Euclidean space R^n and let $\mathcal{T}$ denote the group of all translations of R^n. The Pompeiu problem for the family $\mathcal{S}$ asks when the only function f on R^n such that

$$\int_{\tau(S)} f(x)dx = 0 \text{ for all } \tau \in \mathcal{T} \text{ and } S \in \mathcal{S}$$

is the zero function (1, 2).

We would like to formulate the discrete analog of the Pompeiu problem. The discrete counterpart of R^n is the n-dimensional lattice Z^n, and a subset of Z^n is compact iff it is finite. Because there is no notion of continuity in Z^n, the discrete Pompeiu problem can be stated as follows: Let $\mathcal{S}$ be a family of finite subsets of the n-dimensional lattice Z^n and let $\mathcal{T}$ denote the group of all translations on Z^n. Give necessary and sufficient conditions on the family $\mathcal{S}$ such that

$$\sum_{m \in \tau(S)} f(m) = 0 \text{ for all } \tau \in \mathcal{T}, S \in \mathcal{S} \Rightarrow f \equiv 0. \qquad [1]$$

Note that the system 1 is a system of partial difference equations with constant coefficients. In fact, every finite set S in Z^n defines a partial difference operator $\mathcal{P}_S f(m) = \Sigma_{s \in S} f(m+s)$ and system 1 can be written $\mathcal{P}_S f \equiv 0$ for every $S \in \mathcal{S}$.

In the next section we shall consider general systems of partial difference equations with constant coefficients.

2. Systems of partial difference equations with constant coefficients

Let $\mathcal{F}(Z^n)$ be the set of complex valued functions on Z^n; a *partial difference operator* is a mapping $\mathcal{F}(Z^n) \to \mathcal{F}(Z^n)$ of the form

$$\mathcal{P}f(m) = \sum_{|\alpha| \le N} C_\alpha f(m+\alpha), \qquad [2]$$

where N is an integer, C_α are some complex constants, $m = (m_1, \ldots, m_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ are elements of Z^n, and $|\alpha| = \Sigma_{i=1}^n |\alpha_i|$. The most simple nontrivial partial difference operators are the shifts:

$$X_i f(m) = f(m+e_i), \quad i = 1, \ldots, n,$$

where $e_i = (0, \ldots, 1, 0, \ldots, 0)$ and the 1 is in the ith place. Writing $X^\alpha = X_1^{\alpha_1} \ldots X_n^{\alpha_n}$ we get $X^\alpha f(m) = f(m+\alpha)$, $\alpha \in Z^n$. Thus, [2] can be written

$$\mathcal{P}f(m) = \sum_{|\alpha| \le N} C_\alpha (X^\alpha f)(m) = \Big(\sum_{|\alpha| \le N} C_\alpha X^\alpha\Big) f(m).$$

Hence, $\mathcal{P} = P(X)$, where P is a polynomial in $z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}$—i.e., an element of $\mathcal{A}_n = C[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$, the space, of trigonometric polynomials. Conversely, given a polynomial P, $P(X)$ is a partial difference operator. We shall now define an isomorphism between $\mathcal{F}(Z^n)$ and $\mathcal{A}_n^*$, the dual of $\mathcal{A}_n$.

For every $f{:}Z^n \to C$ define $\hat{f}{:}\mathcal{A}_n \to C$ by

$$\hat{f}\Big(\sum_{k \in Z^n} a_k z^k\Big) = \sum a_k f(k).$$

In particular $\hat{f}(z^m) = f(m)$. Conversely, given a linear functional T on $\mathcal{A}_n$, then $f(m) = T(z^m)$ is a function on Z^n and $T = \hat{f}$. Thus, $f \to \hat{f}$ is a linear isomorphism between $\mathcal{F}(Z^n)$ and $\mathcal{A}_n^*$. We shall see that this simple correspondence is very useful. Let $P(z) = \Sigma_{|\alpha| \le N} C_\alpha z^\alpha$ and suppose that $P(X)f \equiv 0$, then for every $m \in Z^n$

$$0 = P(X)f(m) = \sum_{|\alpha| \le N} C_\alpha f(m+\alpha) = \sum_{|\alpha| \le N} C_\alpha \hat{f}(z^{m+\alpha})$$
$$= \hat{f}\Big(\sum_{|\alpha| \le N} C_\alpha z^{m+\alpha}\Big) = \hat{f}(z^m P(z)).$$

Because $\{z^m\}$ form a basis of $\mathcal{A}_n$, we get $\hat{f}(P(z)u(z)) = 0$ for every $u \in \mathcal{A}_n$ iff $P(X)f \equiv 0$—that is, $P(X)f \equiv 0$ iff $\hat{f}$ annihilates the ideal $P(z)\mathcal{A}_n$. At this stage we can prove our main result.

THEOREM 1. *Let* $P_1, \ldots, P_N$ *be polynomials in* $\mathcal{A}_n$. *Then*

$$P_1(X)f \equiv 0, \ldots, P_N(X)f \equiv 0 \Rightarrow f \equiv 0$$

iff the polynomials $\{P_1(z), \ldots, P_N(z)\}$ *have no common zeros in* $C^n\backslash 0$.

Proof: The assumption about f implies that $\hat{f}$ annihilates $P_1(z)\mathcal{A}_n + \ldots + P_N(z)\mathcal{A}_n$—i.e., $\hat{f}$ annihilates the ideal generated by $\{P_1(z), \ldots, P_N(z)\}$. Because the variety of common zeros of I is empty, it follows from Hilbert's *Nullstellensatz* (ref. 3, p. 157) that the radical of I is $\mathcal{A}_n$. But this implies $I = \mathcal{A}_n$—i.e., $\hat{f}$ is identically zero and consequently $f \equiv 0$.

Conversely, suppose that $\{P_1(z), \ldots, P_N(z)\}$ do have a common zero in $C^n\backslash 0$—i.e., there exists a $z_0 \in C^n\backslash 0$ such that $P_i(z_0) = 0$, $i = 1, \ldots, N$. But then $f(m) = z_0^m$ is a nontrivial solution, because $P_i(X)z_0^m = P_i(z_0)z_0^m = 0$, $i = 1, \ldots, N$.

3. Solution of Pompeiu's problem

Theorem 1 immediately implies the solution of Pompeiu's problem.

THEOREM 2. *For* S *a finite subset of* Z^n *let* $P_S(z) = \Sigma_{s \in S} z^s$. Then $\Sigma_{m \in T(S)} f(m) = 0$ *for every* $S \in \mathcal{S}$, $\tau \in \mathcal{T}$ *iff the polynomials* $\{P_S; S \in \mathcal{S}\}$ *have no common zeros in* C^n.

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The following is a discrete analog of *Theorem 1* in ref. 1.

COROLLARY. *In* Z^2 *let* $\mathcal{S}$ *be the family consisting of three squares of sides* M, N, and K. *Then* $f \equiv 0$ *is the only solution of system* 1 *iff* M + 1, N + 1, K + 1 *are pairwise relatively prime.*

Proof: Let S_M, S_N, and S_K be the squares of sides M, N, and K, respectively. Then

$$P_{S_M} = \sum_{m_1=0}^{M} \sum_{m_2=0}^{M} z_1^{m_1} z_2^{m_2} = (z_1^{M+1} - 1)(z_2^{M+1} - 1)/(z_1 - 1)(z_2 - 1).$$

Similarly, $P_{S_K}(z_1, z_2) = (z_1^{K+1} - 1)(z_2^{K+1} - 1)/(z_1 - 1)(z_2 - 1)$, $P_{S_N}(z_1, z_2) = (z_1^{N+1} - 1)(z_2^{N+1} - 1)/(z_1 - 1)(z_2 - 1)$. Because $(z^r - 1)/(z - 1)$, $(z^l - 1)/(z - 1)$ have no common zeros iff r and l are relatively prime, the statement follows from *Theorem 1.*

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