PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 61, Number 2, December 1976

UNIQUENESS THEOREMS FOR HARMONIC FUNCTIONS OF EXPONENTIAL GROWTH

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ABSTRACT. Two uniqueness theorems for harmonic functions of exponential growth are proved. The first is a generalization to R^n of a theorem proved by Boas [1] for R^2.

1. **Introduction and statement of results.** The purpose of this paper is to prove the following two theorems.

THEOREM A. *Let u be a real-valued harmonic function in R^n satisfying $|u(x)| < Ce^{A|x|}$, where $A < \pi$, $|x| = \sum_{i=1}^n |x_i|$ and C is a constant. If u vanishes on the integer lattice points of the hyperplanes*

$$x_n = 0, \quad \text{and} \quad x_n = a \quad \left(|a| \leqslant (1/(n-1))^{1/2}\right),$$

then it vanishes identically.

THEOREM B. *Let u be as above and suppose both u and $\partial u/\partial x_n$ vanish on the integer lattice points of $x_n = 0$; then u vanishes identically.*

Theorem A is a generalization of a theorem of Boas [1], who proved it for $n = 2$. Boas used the fact that in the two-dimensional case every real-valued entire harmonic function is the real part of an entire (analytic) function. Evidently, this method does not generalize to higher dimensions. Our strategy will be, instead, to view u as a "distribution" (i.e., a continuous linear functional) on the test space of bounded analytic functions on the polystrip $\times_{i=1}^n \{|\text{Im } t_i| < A''\} \subset \mathbf{C}^n$ for $A'' > A$. The referee has kindly informed us that Rao [2] has proved Theorem A by different methods.

2. **Proof of the results.** We shall proceed by a sequence of lemmas.

LEMMA 1. *If u is harmonic in R^n and $|u(x)| < Ce^{A|x|}$, then any partial derivative of u enjoys the same properties.*

PROOF. For any $x_0 \in R^n$ look at the Poisson representation formula for the ball $\|y - x_0\| < 1$, differentiate under the integral sign and estimate.

Let $\mathcal{Q}$ be the class of analytic functions of n complex variables of the form

Received by the editors February 3, 1976.
AMS (MOS) subject classifications (1970). Primary 31B05; Secondary 31B10, 30A98.
Key words and phrases. Harmonic functions, Banach spaces of holomorphic functions, Green's formula.

$$\hat{v}(t) = (2\pi)^{n/2}\int v(x)e^{ixt} \qquad \left(xt = \sum x_i t_i\right),$$

where $v \in C_0^\infty(R^n)$. All these functions are bounded in

$$K_{A''} = \mathop{\times}_{i=1}^{n} \{|\mathrm{Im}\, t_i| < A''\} \quad \text{for every } A''.$$

Define a linear functional on $\mathcal{Q}$ by

$$T_u\left((2\pi)^{n/2}\int_{R^n} v(x)e^{ixt}\right) = \int_{R^n} u(x)v(x). \tag{2.1}$$

The next lemma will show that T_u can be extended continuously to $\mathcal{H}_{A''}$, the Banach space of bounded holomorphic functions on $K_{A''}$, provided $A'' > A$.

LEMMA 2. *Let u be harmonic in R^n and satisfy $|u(x)| < Ce^{A|x|}$. Let $A'' > A$; then T_u defined on $\mathcal{Q}$ by (2.1) can be extended to be a continuous linear functional on the Banach space $\mathcal{H}_{A''}$, which consists of bounded analytic functions on $K_{A''}$ where the norm is given by $\|f\|_{A''} = \sup_{t\in K_{A''}}|f(t)|$.*

PROOF. Let $A < A' < A''$ and let $R_+ = [0, \infty)$, $R_- = (-\infty, 0]$. Then

$$U_{\pm\pm\cdots\pm}(t) = (2\pi)^{-n/2}\int_{R_\pm\times R_\pm\times\cdots\times R_\pm} u(x)e^{-ixt}\,dx$$

belongs to $L^2(\times_{i=1}^n\{\mathrm{Im}\, t_i = \mp A'\})$, and for $\hat{v} \in \mathcal{Q}$,

$$\begin{aligned} T_u(\hat{v}) &= \sum_{\pm}\int_{R_\pm\times\cdots\times R_\pm} u(x)v(x) \\ &= \sum_{\pm}\int_{\times_{i=1}^n\{\mathrm{Im}\, t_i = \mp A'\}} U_{\pm\cdots\pm}(t)\hat{v}(t)\,dt_1\cdots dt_n. \end{aligned} \tag{2.2}$$

The sums in (2.2) each contain 2^n terms, corresponding to all possible choices of sign. Let us consider the term in the sum on the right-hand side of (2.2) involving $U_{--\cdots-}(t)$, and let us write, for the moment, $\Omega = R_- \times R_- \times \cdots \times R_-$. Then, by Green's formula,

$$\begin{aligned} (2\pi)^{n/2}U_{--\cdots-}(t) &= \int_\Omega u(x)e^{-ixt}\,dx = \int_\Omega u(x)\Delta\left(\frac{-e^{-ixt}}{t_1^2+\cdots+t_n^2}\right) \\ &= -\int \Delta u(x)\frac{e^{-ixt}}{t_1^2+\cdots+t_n^2} + \int_{\partial\Omega} u(x)\frac{\partial}{\partial n}\left[\frac{-e^{-ixt}}{t_1^2+\cdots+t_n^2}\right]d\sigma \\ &\quad - \int_{\partial\Omega}\frac{\partial u}{\partial n}\frac{-e^{-ixt}}{t_1^2+\cdots+t_n^2}\,d\sigma. \end{aligned} \tag{2.3}$$

The first term on the right-hand side of (2.3) vanishes since u is harmonic. Now $\partial\Omega$ consists of n pieces: $\partial\Omega = \bigcup_{i=1}^n\{x_i = 0\} \cap \Omega$.

Let us consider the contribution from the face $x_1 = 0$. Here $\partial/\partial n = \partial/\partial x_1$ and

$$\int_{\{x_1=0\}\cap\Omega} u(x)\frac{\partial}{\partial n}\left(\frac{-e^{-ixt}}{t_1^2+\cdots+t_n^2}\right)dx_2\cdots dx_n \tag{2.4}$$
$$=\int_{\{x_1=0\}\cap\Omega} u(0,x_2,\ldots,x_n)\frac{it_1e^{-ixt}\,d\sigma}{t_1^2+\cdots+t_n^2}$$

and

$$\int_{\{x_1=0\}\cap\Omega}\frac{\partial u}{\partial n}(x)\frac{e^{-ixt}\,d\sigma}{t_1^2+\cdots+t_n^2}$$
$$=\int_{\{x_1=0\}\cap\Omega}\frac{\partial u}{\partial x_1}(0,x_2,\ldots,x_n) \tag{2.5}$$
$$-\frac{it_1\exp(-ix_2t_2-\cdots-ix_nt_n)}{t_1^2+\cdots+t_n^2}dx_2\cdots dx_n.$$

Now look at (2.2), the contribution from (2.4) is

$$\int_{\{\operatorname{Im} t_i=A'\}\times \times_2^n\{\operatorname{Im} t_i=A'\}}\hat{v}(t)\,dt_1\cdots dt_n$$
$$\times\int_{\{x_1=0\}\cap\Omega}u(0,x_2,\ldots,x_n)\frac{it_1\exp(-ix_2t_2-\cdots-ix_nt_n)}{t_1^2+\cdots+t_n^2}dx_2\cdots dx_n.$$

But there is a similar contribution, with an opposite sign, from integration on $\{\operatorname{Im} t_1=-A'\}\times\times_2^n\{\operatorname{Im} t_i=A'\}$. Let $\Gamma_{A'}$ be the rectangular contour in the t_1-plane with sides $\pm iA'\pm R$; then, as $R\to\infty$, the sum of these contributions is

$$\int_{\times_2^n\{\operatorname{Im} t_i=A'\}}\phi(t_2,\ldots,t_n)\,dt_2\cdots dt_n\int_{\Gamma_{A'}}\frac{\hat{v}(t_1,\ldots,t_n)it_1}{t_1^2+\cdots+t_n^2}dt_1 \tag{2.6}$$

where

$$\phi(t_2,\ldots,t_n)=\int_{\{x_1=0\}\cap\Omega}u(0,x_2,\ldots,x_n)$$
$$\times\exp(-ix_2t_2-\cdots-ix_nt_n)\,dx_2\cdots dx_n.$$

For fixed $t_2,\ldots,t_n$,

$$\int_{\Gamma_{A'}}\frac{\hat{v}(t_1,\ldots,t_n)it_1}{t_1^2+\cdots+t_n^2}dt_1$$
$$=\begin{cases}\pi i\{\hat{v}(\tau_1,t_2,\ldots,t_n)+\hat{v}(-\tau_1,t_2,\ldots,t_n)\} & \text{if } |\operatorname{Im}\tau_1|<A',\\ 0 \quad \text{if } |\operatorname{Im}\tau_1|>A',\end{cases}$$

where $\tau_1=\tau_1(t_2,\ldots,t_n)$ is given by

$$\tau_1^2+t_2^2+\cdots+t_n^2=0$$

i.e., $\tau_1=i(t_2^2+\cdots+t_n^2)^{1/2}$. Now

$M_{A'}=\{(t_2,\ldots,t_n)\in\mathbf{C}^{n-1},\operatorname{Im} t_2=A',\ldots,\operatorname{Im} t_n=A',|\operatorname{Im}\tau_1|<A'\}$

is seen to be a compact subset of $\times_{i=2}^{n}\{\operatorname{Im} t_i = +A'\}$, and we get that the contribution from the pair of boundary terms (obtained in (2.3)) considered is

$$\pi i \int_{M_{A'}} \phi(t_2, \ldots, t_n)[\hat{v}(\tau_1, t_2, \ldots, t_n) + \hat{v}(-\tau_1, t_2, \ldots, t_n)]\, dt_2 \cdots dt_n, \tag{2.6'}$$

and its absolute value is $\leqslant$ constant $\|\hat{v}\|_{A''}$.

Similarly, if

$$\phi'(t_2, \ldots, t_n) = -\int_{\{x_1=0\}\cap R_- \times \cdots \times R_-} \frac{\partial u}{\partial x_1}(0, x_2, \ldots, x_n) \times \exp(-ix_2t_2 - \cdots - ix_nt_n)\, dx_2 \cdots dx_n,$$

the net contribution from the two terms in (2.2) involving $\phi'(t_2, \ldots, t_n)$ is

$$\int_{\{\operatorname{Im} t_2 = A'\}\times \cdots \times \{\operatorname{Im} t_n = A'\}} \phi'(t_2, \ldots, t_n) \int_{\Gamma_{A'}} \frac{\hat{v}(t_1, \ldots, t_n)}{t_1^2 + \cdots + t_n^2}\, dt_1$$

which is equal to

$$\pi \int_{M_{A'}} \phi'(t_2, \ldots, t_n) \cdot \frac{1}{\tau_1}[\hat{v}(\tau_1, t_2, \ldots, t_n) - \hat{v}(-\tau_1, t_2, \ldots, t_n)]\, dt_2 \cdots dt_n \tag{2.7'}$$

which, in absolute value, is $\leqslant$ constant $\|v\|_{A''}$.

In a similar way we can consider all other terms of (2.2) and write it as a sum of $n2^{n-1}$ terms of the form (2.6′) and $n2^{n-1}$ terms of the form (2.7′). The resulting formula defines $T_u(f)$ for every $f \in \mathcal{H}_{A''}$ and T_u is a bounded linear functional on $\mathcal{H}_{A''}$.

LEMMA. 3. *For every $x \in R^n$, $T_u(e^{ixt}) = (2\pi)^{-n/2}u(x)$.*

PROOF. Let K_ε be a C^∞ compact support approximate identity; then $\int K_\varepsilon(y - x)e^{iyt}\, dt \to e^{ixt}$ in the topology of $\mathcal{H}_{A''}$ and

$$T_u(e^{ixt}) = \lim_{\varepsilon\to 0} T_u\left(\int K_\varepsilon(y - x)e^{iyt}\, dy\right)$$

$$= \lim_{\varepsilon\to 0}(2\pi)^{-n/2}\int K_\varepsilon(y - x)u(y) = (2\pi)^{-n/2}u(x).$$

LEMMA 4. *There exist measures $d\mu_1, d\mu_2$ on $\{t_n = 0\} = \mathbf{C}^{n-1}$ supported in the compact set*

$$L_{A''} = \left\{(t_1, \ldots, t_{n-1});\ |\operatorname{Im} t_1| < A'', \ldots, |\operatorname{Im} t_{n-1}| < A'',\ \left|\operatorname{Re}(t_1^2 + \cdots + t_{n-1}^2)^{1/2}\right| < A''\right\}$$

such that for every $f \in \mathcal{H}_{A''}$,

$$T_u(f) = \int f\left(t_1, \ldots, t_{n-1}, i\left(t_1^2 + \cdots + t_{n-1}^2\right)^{1/2}\right) d\mu_1 \tag{2.8}$$
$$+ \int f\left(t_1, \ldots, t_{n-1}, -i\left(t_1^2 + \cdots + t_{n-1}^2\right)^{1/2}\right) d\mu_2.$$

In particular,

$$u(x) = \int \exp(ix_1t_1 + \cdots + ix_{n-1}t_{n-1})\exp\left(-\left(t_1^2 + \cdots + t_{n-1}^2\right)^{1/2}x_n\right) d\mu_1 \tag{2.9}$$
$$+ \int \exp(ix_1t_1 + \cdots + ix_{n-1}t_{n-1})\exp\left(\left(t_1^2 + \cdots + t_{n-1}^2\right)^{1/2}x_n\right) d\mu_2.$$

PROOF. Let $V_{A''} = \{(t_1, \ldots, t_n) \in K_{A''};\ t_1^2 + t_2^2 + \cdots + t_n^2 = 0\}$. Then by the proof of Lemma 2, by adding all the terms like (2.6′) and (2.7′), we get that there exists a measure $d\nu$, supported in $V_{A''}$ such that for every $f \in \mathcal{H}_{A''}$, $T_u(f) = \int f\, d\nu$.

Let $d\nu = d\nu_1 + d\nu_2$, where $d\nu_1$ is supported in

$$\left\{\left(t_1, t_2, \ldots, i\left(t_1^2 + \cdots + t_{n-1}^2\right)^{1/2}\right)\right\}$$

and $d\nu_2$ is supported in

$$\left\{\left(t_1, t_2, \ldots, t_{n-1}, -i\left(t_1^2 + \cdots + t_{n-1}^2\right)^{1/2}\right)\right\}.$$

Let $d\mu_1$, $d\mu_2$ be the projections of $d\nu_1$, $d\nu_2$, respectively, on $t_n = 0$. Then the lemma follows since $d\mu_1$, $d\mu_2$ are supported in the projection of $V_{A''}$ on $t_n = 0$ which is $L_{A''}$.

Now we are in a position to prove the theorems.

PROOF OF THEOREM A. Since $A < \pi$ we can choose $A < A'' < \pi$. It is easily seen that $L_{A''}$ is contained in $\times_{i=1}^{n-1}\{|\mathrm{Im}\, t_i| < A''\} \times \{|\mathrm{Re}\, t_i| < A''\}$, and since $A'' < \pi$, the span of $\{e^{ixt};\ x \in Z^{n-1}\}$, where Z^{n-1} are the integer lattice points of R^{n-1}, is dense in the space of bounded holomorphic functions on $L_{A''}$. By (2.9), $d\mu_1 + d\mu_2 \equiv 0$ and

$$\exp\left(-a\left(t_1^2 + \cdots + t_{n-1}^2\right)^{1/2}\right) d\mu_1 + \exp\left(a\left(t_1^2 + \cdots + t_{n-1}^2\right)^{1/2}\right) d\mu_2 \equiv 0.$$

Since $a \leqslant (1/(n-1))^{1/2}$ it follows that $d\mu_1$, $d\mu_2$ are supported in $\{t_1^2 + \cdots + t_{n-1}^2 = 0\}$ and by (2.9), $u(x)$ is identically zero. □

PROOF OF THEOREM B. Applying $\partial/\partial x_n$ to (2.9) we get

$$\frac{\partial u}{\partial x_n}(x_1, \ldots, x_{n-1}, 0)$$
$$= -\int \exp(ix_1t_1 + \cdots + ix_{n-1}t_{n-1})\left(t_1^2 + \cdots + t_{n-1}\right)^{1/2} d\mu_1$$
$$+ \int \exp(ix_1t_1 + \cdots + ix_{n-1}t_{n-1})\left(t_1^2 + \cdots + t_{n-1}^2\right)^{1/2} d\mu_2.$$

As in the proof of the Theorem A we get that

$$d\mu_1 + d\mu_2 \equiv 0, \qquad \left(t_1^2 + \cdots + t_{n-1}^2\right)^{1/2}(d\mu_1 - d\mu_2) \equiv 0.$$

Thus $d\mu_1 = -d\mu_2$ is supported at the set $\{t_1^2 + \cdots + t_{n-1}^2 = 0\}$, and by using (2.9) it once again follows that u vanishes identically.

ACKNOWLEDGEMENT. The author wishes to thank Professor Harry Dym, Dr. Amy Novick, and Dr. Akiva Gabor for helpful comments and discussions.

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