A NOTE FOR DORON: THE GENERATING FUNCTION FOR TOTAL DISPLACEMENT (SPEARMAN'S FOOTRULE) AND INVERSION NUMBER

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Let S_n denote the symmetric group of permutations of $\{1, 2, \ldots, n\}$. Diaconis and Graham [1] studied what they called *Spearman's disarray* for permutations in S_n. The disarray statistic, which Knuth calls *total displacement* [4, Problem 5.1.1.28], is defined for a permutation w as

$$\sum_{i=1}^{n} |w(i) - i| = 2 \sum_{w(i)>i} (w(i) - i).$$

In work of Petersen and Tenner [5], half of the total displacement is shown to be equal to the *depth* of a permutation, which is a measure of certain minimal factorizations of a permutation into transpositions. That is, define

$$\mathrm{dep}(w) = \min\left\{ \sum_{s=1}^{k} (j_s - i_s) : w = (i_1\, j_1) \cdots (i_k\, j_k) \right\}. \tag{0.1}$$

Then [5, Theorem 1.1] shows that

$$\mathrm{dep}(w) = \sum_{w(i)>i} (w(i) - i). \tag{0.2}$$

In Guay-Paquet and Petersen [3], the authors developed the generating function for depth, via a map from permutations to Motzkin paths that takes depth to area. Zeilberger has asked [6] whether this approach can be adapted to study other statistics, like Spearman's ρ, or the joint distribution of disarray with inversion number. I don't know about $\rho = \sum_{i=1}^{n} (w(i) - i)^2$, but keeping track of inversions seems to be a very natural q-analogue.

The map from [3] is $\phi\colon S_n \to \mathrm{Motz}_n$ by $\phi(w) = p_1 \cdots p_n$, where

$$p_i = \begin{cases} U & \text{if } w^{-1}(i) > i < w(i), \\ D & \text{if } w^{-1}(i) < i > w(i), \\ H & \text{otherwise.} \end{cases}$$

For example, if $w = 3715246$, we have $\phi(w) = UUDUDHD$.

Here are two results from [3].

Proposition 0.1. *Let $w \in S_n$ be a permutation and $\phi(w) = p_1 \cdots p_n$ be the associated Motzkin path. Then,*

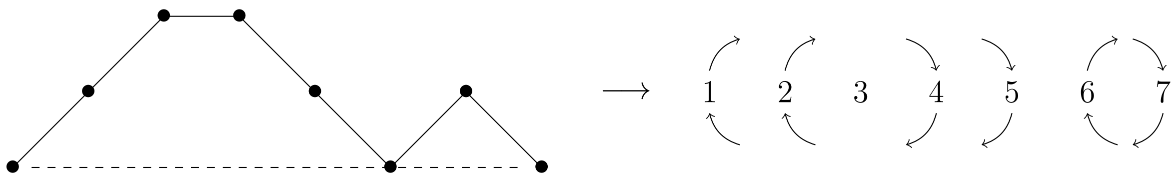
$$\mathrm{dep}(w) = \sum_{p_k = D} k - \sum_{p_i = U} i.$$

Proposition 0.2. *For any $w \in S_n$,* $\mathrm{dep}(w) = \mathrm{area}(\phi(w))$.

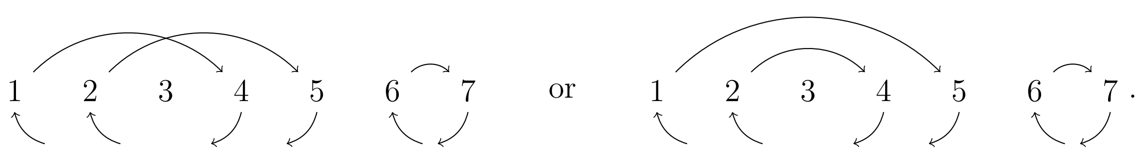
Given that the map $\phi\colon S_n \to \mathrm{Motz}_n$ encodes the depth of a permutation as the area of the associated Motzkin path, the next step in determining the distribution of the depth statistic is to compute the distribution of inversions on the preimage $\phi^{-1}(p)$ for each Motzkin path p.

This is demonstrated with the following example that suggests a conjecture. Might be nice for an interested student to explain and prove.

Example 0.3. Given the Motzkin path $p = UUHDDUD$, we start by writing the numbers $1, \ldots, 7$ on a line and drawing incoming and outgoing half-arcs for every U and D (without connecting the half-arcs to each other):



When drawing a permutation w such that $\phi(w) = p$, each excedance of w is drawn as a right-pointing arrow above the line of numbers, and these arrows can be grouped into strings of right-pointing arrows, which start at a position i with $p_i = U$ and end at a position k with $p_k = D$, possibly with intermediate steps at positions j with $p_j = H$. So, to form the permutations that correspond to this path, we will first ignore the positions j with $p_j = H$ and match up all the half-arcs above the line of numbers, outgoing with incoming, to indicate the starting and ending positions of each string of right-pointing arrows. In this example, we are forced to form the pair $6 \to 7$, but we have two choices for pairing the outgoing half-arcs $1 \to \cdot$, $2 \to \cdot$ and the incoming half-arcs $\cdot \to 4$, $\cdot \to 5$; that is:



Independently, we also have two choices for matching up the half-arcs below the line of numbers into (for now, single-step) strings of left-pointing arrows in the permutation diagram for w. Finally, to complete the diagram, we only need to decide what to do with the position 3, for which $p_3 = H$: it can either be a fixed point of w, or join one of the two strings of right-pointing arrows above it, or join one of the two strings of left-pointing arrows below it, for a total of five choices. (Note that for all matchings here, there are two strings of right-pointing arrows above position 3 and two strings of left-pointing arrows below it.) All in all, we have $2 \cdot 2 \cdot 5 = 20$ possible diagrams (all valid) for a permutation w such that $\phi(w) = p = UUHDDUD$.

It turns out that the distribution of inversions for the 20 permutations formed in this way is:

$$q^7(1+q)^2(2+2q+q^2) = q^7[2]^2([2]+[3]),$$

where $[n]$ is shorthand for the q-integer $1 + q + \cdots + q^{n-1}$.

In general, we can count the number of permutations corresponding to a given Motzkin path by reconstructing the possible permutation diagrams as in Example 0.3; first we count the number of ways of matching up the outgoing and incoming half-arcs above the line of numbers, then we count the number of ways of matching up the half-arcs below, and finally we count the number of ways of dealing with the positions corresponding to H steps. This is described in [3], but we now need to keep track of inversions too. Here is the proposed way to do this.

First, we will need to define the *height* of each step in a Motzkin path. We draw our paths starting at $(0,0)$ and define the height h_i of a step p_i to be the maximum height achieved on that part of the path. That is, $h_i = j$ if

- $p_i = U$ from $(i-1, j-1)$ to (i,j),
- $p_i = H$ from $(i-1, j)$ to (i,j), or
- $p_i = D$ from $(i-1, j)$ to $(i, j-1)$.

For example, the steps of $p = UUHDDUD$ have heights $(h_1, \ldots, h_7) = (1,2,2,2,1,1,1)$.

We define the *weight* of step p_i to be 1 if $h_i = 0$, but if $h_i > 0$ we have

$$\omega_i = \begin{cases} [h_i](qt)^{(2h_i-1)/2} & \text{if } p_i = U \text{ or } p_i = D, \\ ([h_i] + [h_i+1])(qt)^{h_i} & \text{if } p_i = H, \end{cases}$$

and the weight of a path $p = p_1 \cdots p_n$ to be the product

$$\omega(p) = \omega_1 \cdots \omega_n \,.$$

Then, the weight of a Motzkin path p is the generating function for inversions and depth of permutations in its preimage $\phi^{-1}(p)$.

Conjecture 1. *Let $p \in \mathrm{Motz}_n$. Then*

$$\omega(p) = \sum_{w \in \phi^{-1}(p)} q^{\mathrm{inv}(w)} t^{\mathrm{dep}(w)} = t^{\mathrm{area}(p)} \sum_{w \in \phi^{-1}(p)} q^{\mathrm{inv}(w)}.$$

In the example of $p = UUHDDUD$,

$$\begin{aligned} \omega(p) &= (qt)^{1/2} \cdot [2](qt)^{3/2} \cdot ([2]+[3])(qt)^2 \cdot [2](qt)^{3/2} \cdot (qt)^{1/2} \cdot (qt)^{1/2} \cdot (qt)^{1/2}, \\ &= (qt)^7 [2]^2 ([2]+[3]). \end{aligned}$$

If Conjecture 1 is true, we can follow [3] to express the generating function for permutations with respect to depth and inversions as

$$F(q,t,z) = \sum_{n \geq 0} \sum_{w \in S_n} q^{\mathrm{inv}(w)} t^{\mathrm{dep}(w)} z^n = \sum_{p \in \mathrm{Motz}} \omega(p) z^{|p|},$$

where $|p|$ is the number of steps in the path p. The continued fraction for Motzkin paths counted by the product of steps weighted according to height is:

$$F = \cfrac{1}{1 - b_0 - \cfrac{a_1 c_1}{1 - b_1 - \cfrac{a_2 c_2}{1 - b_2 - \cfrac{a_3 c_3}{1 - \cdots}}}},$$

where b_h is the weight of a horizontal step at height h, a_h is the weight of step U at height h, and c_h is the weight of step D at height h.

If we substitute the weights for $a_h = c_h = [h](qt)^{(2h-1)/2}z$, $b_h = ([h]+[h+1])(qt)^h z$, we get $a_h c_h = [h]^2(qt)^{(2h-1)}z^2$, and so

(0.3)

$$F(q,t,z) = \cfrac{1}{1 - z - \cfrac{qtz^2}{1 - ([1]+[2])qtz - \cfrac{[2]^2(qt)^3z^2}{1 - ([2]+[3])(qt)^2z - \cfrac{[3]^2(qt)^5z^2}{1 - ([3]+[4])(qt)^3z - \cfrac{[4]^2(qt)^7z^2}{1 - \cdots}}}}}.$$

I think (from Chapter 5 of [2]) this expression is equivalent to the following:

(0.4)

$$F(q,t,z) = \cfrac{1}{1 - \cfrac{z}{1 - \cfrac{qtz}{1 - \cfrac{[2]qtz}{1 - \cfrac{[2](qt)^2z}{1 - \cfrac{[3](qt)^2z}{1 - \cfrac{[3](qt)^3z}{1 - \cfrac{[4](qt)^3z}{1 - \cfrac{[4](qt)^4z}{1 - \cdots}}}}}}}}.$$

Here, for $k \geq 0$, the $(2k)$th term is $[k+1](qt)^k z$, and the $(2k+1)$st term is $[k+1](qt)^{k+1}z$.

References

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