A note on Kyle Petersen’s conjecture ([3])

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It is well known (cf. [1]) that the Fibonacci polynomials $F_n(x, s) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-j}{j} s^j x^{n-2j}$ satisfy

$$F_n(x + y, -xy) = \frac{x^n - y^n}{x - y}$$

and the Lucas polynomials $L_n(x, s) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-j}{j} s^j x^{n-2j}$ with $L_0(x, s) = 2$ satisfy

$$L_n(x + y, -xy) = x^n + y^n.$$

Consider the Kyle map $K(q) = \sum_{j} (-1)^j q^j (1 + q)^{n-2j}$ which (as Richard Stanley [2] observed) satisfies $K(f(q)g(q)) = K(f(q))K(g(q))$.

Since $K\left(\frac{1 - q^n}{1 - q}\right) = K(F_n(1 + q, -q)) = F_n(1, 1) = F_n$, the above mentioned result

$$K([n]!) = \prod_{j=1}^{n} F_j$$

follows.

In the same way $K(1 + q^n) = K(L_n(1 + q, -q)) = L_n(1, 1) = L_n$ where

$$(L_n) = (2, 1, 3, 4, 7, 11, 18, \ldots)$$

is the sequence of Lucas numbers.

Thus $K\left(\prod_{j=1}^{n} (1 + q^j)\right) = \prod_{j=1}^{n} L_j$.

References


[2] R. Stanley, Email message ,
http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/kylefb.html