Dennis Stanton[1], June 16, 2016

We prove the first two super-congruences conjectured by Moa Apagodu and Doron Zeilberger in the paper

"Using the "Freshman's Dream" to Prove Combinatorial Congruences."

(See `http://arxiv.org/abs/1606.03351`)

1. Super-Conjecture 1

Super-Conjecture 1.1. *Let $p \geq 5$ be a prime, and*

$$A = \sum_{n=0}^{p-1} \binom{2n}{n}.$$

Then

$$\begin{aligned} A &\equiv 1 \mod p^2 \text{ if } p \equiv 1 \mod 3, \\ A &\equiv -1 \mod p^2 \text{ if } p \equiv 2 \mod 3. \end{aligned}$$

Proof. Already from the paper we know

$$A = CT \frac{(1+x)^{2p}}{x^{p-1}(1+x+x^2)} = CT \frac{(1-x)(1+x)^{2p}}{x^{p-1}(1-x^3)}.$$

So

$$A = \sum_{k \geq 0} \left(\binom{2p}{p-1-3k} - \binom{2p}{p-2-3k} \right) \tag{1}$$

CASE 1 $p \equiv 1 \mod 3$. In this case, the $k = (p-1)/3$ term in (1) is 1, and all of the other terms are divisible by p,

$$A = 1 + pB$$

where

$$B = \frac{1}{p} \sum_{0 \leq k < (p-1)/3} \left(\binom{2p}{p-1-3k} - \binom{2p}{p-2-3k} \right).$$

We'll show that the integer B satsifies $B \equiv 0 \mod p$, so that $A \equiv 1 \mod p^2$.

Since

$$\begin{aligned} \frac{1}{p} \binom{2p}{p-1-3k} &\equiv 2 \frac{(p-1)!}{(p-1-3k)!(3k+1)!} \mod p, \\ \frac{1}{p} \binom{2p}{p-2-3k} &\equiv 2 \frac{(p-1)!}{(p-2-3k)!(3k+2)!} \mod p \end{aligned}$$

we see that

$$B \equiv 2 \sum_{0 \leq k < (p-1)/3} \left(\frac{(p-1)!}{(p-1-3k)!(3k+1)!} - \frac{(p-1)!}{(p-2-3k)!(3k+2)!} \right) \mod p$$

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However, by trisecting the binomial theorem this sum of integers is 0

$$\sum_{0\le k<(p-1)/3}\left(\frac{(p-1)!}{(p-1-3k)!(3k+1)!}-\frac{(p-1)!}{(p-2-3k)!(3k+2)!}\right)=0,$$

so

$$B\equiv 0 \mod p.$$

<u>CASE 2</u> $p\equiv 2 \mod 3$. In this case the second term with $k=(p-2)/3$ gives -1. We proceed as in CASE 1, the trisection identity we need is

$$\sum_{0\le k\le(p-2)/3}\frac{(p-1)!}{(p-1-3k)!(3k+1)!}-\sum_{0\le k\le(p-5)/3}\frac{(p-1)!}{(p-2-3k)!(3k+2)!}=0,$$

□

2. Super-Conjecture 1"

Super-Conjecture 2.1. *Let $p\ge 5$ be a prime, r be a positive integer, and*

$$A=\sum_{n=0}^{rp-1}\binom{2n}{n}.$$

Then

$$A\equiv \alpha_r \mod p^2 \text{ if } p\equiv 1 \mod 3,$$
$$A\equiv -\alpha_r \mod p^2 \text{ if } p\equiv 2 \mod 3.$$

where

$$\alpha_r=\sum_{n=0}^{r-1}\binom{2n}{n}.$$

Proof. (sketch, assuming a fact (3) you probably know, if true) This time

$$A=CT\frac{(1+x)^{2rp}}{x^{rp-1}(1+x+x^2)}=CT\frac{(1-x)(1+x)^{2rp}}{x^{rp-1}(1-x^3)}.$$

So

$$A=\sum_{k\ge 0}\left(\binom{2rp}{rp-1-3k}-\binom{2rp}{pr-2-3k}\right) \tag{2}$$

As before, we separate the terms which have a p for sure, namely those whose binomial denominator is not a multiple of p from those whose binomial denominator is a multiple of p. The non-multiples of p will sum to 0 as before, and we are left with the extra terms.

<u>CASE 1</u> $p\equiv 1 \mod 3$. Here the terms with denominator parameter a multiple of p are

$$EXTRA=\sum_{j\ge 0}\left(\binom{2rp}{(r-1-3j)p}-\binom{2rp}{(r-2-3j)p}\right).$$

Assuming

$$\binom{Dp}{Ep}\equiv\binom{D}{E} \mod p^2 \tag{3}$$

then

$$EXTRA \equiv \sum_{j\geq 0} \left(\binom{2r}{r-1-3j} - \binom{2r}{r-2-3j} \right) = \alpha_r \mod p^2$$

by (1), which works for any positive integer p.

<u>CASE 2</u> $p \equiv 2 \mod 3$. Here the terms with denominator parameter being a multiple of p are

$$EXTRA = \sum_{j\geq 0} \left(\binom{2rp}{(r-2-3j)p} - \binom{2rp}{(r-1-3j)p} \right).$$

then

$$EXTRA \equiv \sum_{j\geq 0} \left(\binom{2r}{r-2-3j} - \binom{2r}{r-1-3j} \right) = -\alpha_r \mod p^2$$

again by (1).

□

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