AN INTEGRAL IDENTITY

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1. A simple proof

We are going to see that a family of integral identities including that of [EZZ19] admit a quite simple proof. Although the proof in [EZZ19] is elementary it requires quite a number of calculations that become trivial for a computer. Our proof is "more human" in the old-fashioned mathematical sense.

Proposition 1.1. *For $n, k \in \mathbb{Z}_{\geq 0}$ and $a, b > 0$ we have*

$$\int_0^1 \frac{x^n(1-x)^n}{(x+a)^{k+1}(x+b)^{n+1}}\,dx = \int_0^1 \frac{x^n(1-x)^n(x+b)^{k-n}}{\big((a-b)x+(a+1)b\big)^{k+1}}\,dx.$$

Proof. With the change of variables $x \mapsto b(1-x)/(x+b)$ we have

$$\int_0^1 \frac{x^n(1-x)^n}{(x+b)^{n+k+1}}\,dx = \frac{1}{b^k(b+1)^k}\int_0^1 \frac{x^n(1-x)^n}{(x+b)^{n-k+1}}\,dx. \tag{1.1}$$

For each fixed b consider the function

$$F(a) = \int_0^1 \frac{x^n(1-x)^n}{(x+b)^{n+1}}\Big(\frac{1}{x+a} - \frac{x+b}{a(x+b)+b(1-x)}\Big)\,dx.$$

By (1.1), $F^{(k-1)}(b) = 0$ for $k \geq 1$ and as F is analytic, it is identically zero. The identity is equivalent to $F^{(k)} = 0$. □

Remark. The same argument gives the result for $n \in \mathbb{R}_{>-1}$. The numerical calculations suggest that it also holds for any $k \in \mathbb{R}$ but the proof does not give this case. Perhaps it can be got from a version of Carlson's theorem.

2. Some consequences

Before finding the change of variables leading to (1.1), a first attempt to get it was to employ combinatorial identities for the coefficients in the Laurent expansion but they turned out to be quite involved. One can proceed the other way around to get combinatorial consequences.

Corollary 2.1. *The function $f_n(x) = x^n(1+x)^n$ satisfies*

$$f_k f_n^{(n+k)} = \frac{(n+k)!}{(n-k)!} f_n^{(n-k)} \qquad \text{for } 0 \leq k \leq n.$$

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Essentially comparing coefficients one gets a triple binomial identity that surely can be obtained using a WZ pair.

Corollary 2.2. *For each k, l, n nonnegative integers with $l, k \le n$*

$$\sum_{m=0}^{k} \binom{k}{m}\binom{n}{m+l}\binom{2m+2l}{k+n} = \binom{n}{l}\binom{2l}{n-k}.$$

The definition of Legendre polynomial assure that they are eigenfunctions of the differential operator $\frac{d}{dx}(x^2-1)\frac{d}{dx}$. Although $\frac{d}{dx}$ and (x^2-1) do not commute and apparently there is not a simple formula for the commutator of their powers, Legendre polynomials are eigenfunctions of a simple operator composed by powers of these operators.

Corollary 2.3. *Let P_n be the n-th Legendre polynomial. Then for $0 \le k \le n$*

$$L[P_n] = \frac{(n+k)!}{(n-k)!}P_n \qquad \textit{where} \quad L = \frac{d^k}{dx^k}(x^2-1)^k\frac{d^k}{dx^k}.$$

Proof of Corollary 2.1. By Taylor expansion at $x=-b$ we have

$$x^n(1-x)^n = (-1)^n f_n(-x) = \sum_{m=0}^{2n} c_m(x+b)^m \qquad \text{with} \quad c_m = (-1)^{n+m}\frac{f_n^{(m)}(b)}{m!}.$$

If we substitute this in (1.1) (or in Proposition 1.1 with $a=b$) a term with $\log(b+1)-\log b$ appears in the LHS for $m=n+k$ (the rest are rationals) and in the integral of the RHS for $m=n-k$. Hence $c_{n+k} = c_{n-k}/\big(b(b+1)\big)^k$. □

Proof of Corollary 2.2. In Corollary 2.1 change the variable $x \mapsto (x-1)/2$ to obtain, clearing denominators,

$$(n-k)!(x^2-1)^k\frac{d^{n+k}}{dx^{n+k}}(x^2-1)^n = (n+k)!\frac{d^{n-k}}{dx^{n-k}}(x^2-1)^n. \tag{2.1}$$

The left hand side is

$$(n-k)!\sum_{l_1}(-1)^{k-l_1}\binom{k}{k-l_1}x^{2l_1}\cdot(n+k)!\sum_{l_2}(-1)^{n-l_2}\binom{n}{l_2}\binom{2l_2}{n+k}x^{2l_2-n-k}.$$

And the right hand side is

$$(n+k)!(n-k)!\sum_{l}(-1)^{n-l}\binom{n}{l}\binom{2l}{n-k}x^{2l-n+k}.$$

Comparing the coefficients renaming $l_1 = k-m$ and $l_2 = m+l$ we obtain the result. □

We have learned that (2.1) is a known identity expressing a symmetry of the associated Legendre polynomials. In [Wik19] this is obtained from Rodrigues' formula and the general Legendre equation. In [Wes] there is a simple elementary proof. The one obtained here (Corollary 2.1) is competitively simple.

Proof of Corollary 2.3. Recalling that P_n is proportional to $\frac{d^n}{dx^n}(x^2-1)^n$ (Rodrigues' formula) this follows taking the k-derivative of (2.1). $\square$

Actually Corollary 2.3 also implies Corollary 2.1 by repeated integration and noting that both sides in Corollary 2.1 are divisible by x^k.

References

[EZZ19] S. B. Ekhad, D. Zeilberger, and W. Zudilin. Two Definite Integrals That Are Definitely (and Surprisingly!) Equal. arXiv:1911.01423 [math.CA], 2019.

[Wes] D. B. Westra. Identities and properties for associated Legendre functions. `https://www.mat.univie.ac.at/~westra/associatedlegendrefunctions.pdf`.

[Wik19] Wikipedia contributors. Associated legendre polynomials — Wikipedia, the free encyclopedia, 2019. [Online; accessed 25-November-2019].