On two constants in a paper by R. Dougherty-Bliss, C. Koutschan and D. Zeilberger

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Abstract

We simplify the expression of two constants occurring in a paper by Dougherty-Bliss, Koutschan and Zeilberger. This shows that both constant are transcendental, thanks to a deep result of Nesterenko.

1 Introduction

The famous irrationality proofs for $\zeta(3)$ by Apéry [1] and by [2] inspired the paper [4], where Dougherty-Bliss, Koutschan and Zeilberger search "miraculous" irrationality proofs à la Apéry. Two of the constants that their paper mentions (on top of p. 987) are

$$W_1 := -24 - 81\sqrt{\pi}\frac{\Gamma(7/3)}{\Gamma(-1/6)} \quad \text{and} \quad W_2 := \frac{13}{2} - \frac{6\Gamma(19/6)}{\sqrt{\pi}\Gamma(8/3)}.$$

We will confirm that these two constants are transcendental thanks to a theorem of Nesterenko.

2 Rewriting the two constants

It is probably well known that $\Gamma(1/6) = 3^{1/2}2^{-1/3}\pi^{-1/2}\Gamma(1/3)^2$ (see, e.g., [5]). This expression can be obtained by taking $x = 1/6$ after combining the duplication and the reflection formulas for the gamma function:

$$\Gamma(2x) = 2^{2x-1}\pi^{-1/2}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = 2^{2x-1}\pi^{-1/2}\Gamma(x)\frac{\pi}{\sin(\pi(x+\frac{1}{2}))\Gamma(\frac{1}{2}-x)}.$$

The patient reader can now prove the following equalities, by repeatedly using the identity $\Gamma(x+1) = x\Gamma(x)$:

$$W_1 = -24 + 3^{3/2}2^{-1/3}\left(\frac{\Gamma(1/3)^3}{\pi}\right) \quad \text{and} \quad W_2 = \frac{13}{2} - \frac{273}{80}\left(\frac{\Gamma(1/3)^3 2^{1/3}}{\pi^2}\right).$$

3 W_1 and W_2 are transcendental

A deep theorem of Nesterenko (Corollary 5 in [6, p. 1321]) states that $\pi, e^{\pi\sqrt{3}}$ *and* $\Gamma(1/3)$ *are algebraically independent on the rationals.* This immediately implies the following result for the (rewritten) constants:

W_1 and W_2 are transcendental.

(Note that a result weaker than Nesterenko's, namely that π and $\Gamma(1/3)$ are algebraically independent suffices here: this was proved by Čudnovs'kiĭ, see [3]; also see [8].)

References

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