Recurrences for the two-dimensional Ising model

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In 1971 when I was a post-doc at King's College, University of London, Martin Sykes mentioned to me that computationally, the most precise and rapid way to obtain the coefficients of the spontaneous magnetisation of the Ising model was from the linear recurrence relation satisfied by the coefficients. The spontaneous magnetisation for the square lattice is

$$\frac{I}{m} = \left(1 - \frac{16u^2}{(1-u)^4}\right)^{\frac{1}{8}} = \sum_{r=0}^{\infty} b_r u^r,$$

where $u = \exp -(4/Jk_BT)$. Then

$$nb_n = 6(n+1)b_{n+1} + 4b_{n+2} - 6(n+3)b_{n+3} + (n+4)b_{n+4},$$

with $b_0 = 1$, $b_1 = 0$, $b_2 = -2$, $b_3 = -8$.
This recurrence (with n shifted) is given in the footnote to Table X on page 420 in the article on the Ising model by C. Domb in *Phase Transitions and Critical Phenomena, Vol 3.* eds. C. Domb and M.S. Green, Academic Press 1974.

We laughed and said that if we had the coefficients in the 1940s that we had now (in the 1970s) we could have found the solution experimentally (or even 20 years earlier, based on the coefficients known at that time). We speculated whether this was perhaps also true for the free-energy?

Together with my room-mate, Geoff Joyce, we considered this question. We immediately realised that the Onsager solution for the internal energy was usually written in terms of the complete elliptic integral of the first kind, $\mathbf{K}(z)$. This was known to be D-finite (we didn't know this term at the time, but knew that the coefficients of the complete elliptic integral satisfied a linear recurrence). To be precise, the internal energy $U(u)$ satisfies

$$U(u) = \frac{1+u}{1-u} + \frac{2(1-6u+u^2)}{\pi(1-u)(1+u)}\mathbf{K}\left(\frac{4\sqrt{u}(1-u)}{(1+u)^2}\right) = \sum_{i=0}^{\infty} c_n u^n.$$

The coefficients satisfy the recurrence

$$\begin{aligned}(4n^5+36n^4+97n^3+75n^2)c_n &= (28n^5+296n^4+1087n^3+1702n^2+1133n+230)c_{n+1}\\ &-(24n^5+240n^4+790n^3+968n^2+338n-60)c_{n+2}-(24n^5+360n^4+1990n^3+4882n^2+4908n+1300)c_{n+3}\\ &+(28n^5+404n^4+2167n^3+5203n^2+5138n+1260)c_{n+4}-(4n^5+64n^4+377n^3+980n^2+1025n+250)c_{n+5},\end{aligned}$$

with $c_0 = 2$, $c_1 = 0$, $c_2 = -8$, $c_3 = -24$, and $c_4 = -72$. (Geoff Joyce in 1971 first wrote down an equivalent, but differently normalised recurrence relation, and corresponding differential equation).

This made me realise that searching for such recurrences (equivalent to fitting to a D-finite ODE) represented a powerful new method of series analysis.

Joyce and I wrote this up in 1972 in *On a new method of series analysis in lattice statistics*, J.Phys A:Gen. Phys. **5**, L81, 1972 (G.S. Joyce and A. J. Guttmann).

Application of the method to a number of test series, and the explicit connection with Onsager's solution was then discussed in my paper *On the recurrence relation method of series analysis*, J.Phys A:Gen. Phys. **8**, 1081, 1975 (A. J. Guttmann).

Both Joyce and I gave seminars on this at the time, and pointed out that we could have conjectured Onsager's solution based on the known series coefficients at that time.

This history, in an attenuated form, is also given in my article *Asymptotic analysis of power series expansions* in *Phase Transitions and Critical Phenomena, Vol 13.* eds. C. Domb and J. Lebowitz, Academic Press 1989 on pages 83-84.

This method of series analysis we called the recurrence-relation method, as it involved finding linear recurrences between coefficients. It was soon re-named the method of differential approximants, as it became clear that the underlying differential equation was a more natural object. It has remained the most effective method for extracting the exact or approximate asymptotics of coefficients with the usual power-law behaviour, $d_n \sim C \cdot \mu^n \cdot n^g$.

Of course this is not the same as what you have done, where you've made clever use of symmetries, and chosen an appropriate natural variable, but it's philosophically along the same lines, and I thought you might be interested to know what was done on this topic some 46 years ago.