# von Neumann Poker with Finite Decks


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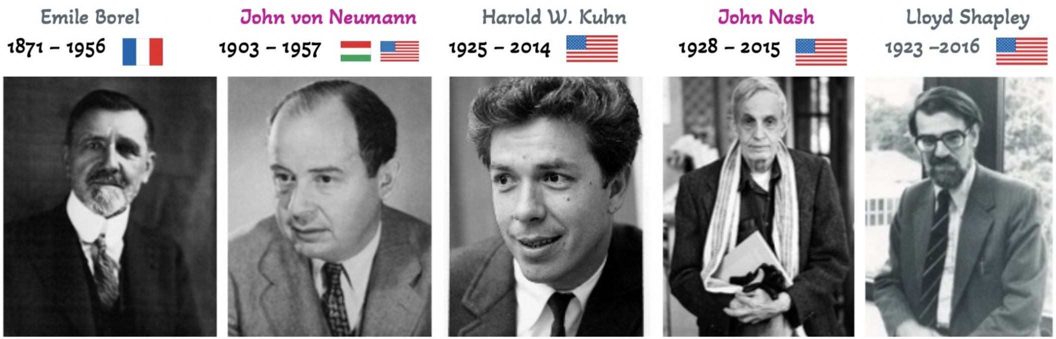
# The World of Poker x Math

- Pioneer mathematicians in Poker include Émile Borel, John von Neumann, Harold W. Kuhn, John Nash, and Lloyd Shapley.
- They believed that real-life scenarios mirror poker with their elements of bluffing and strategic thinking.
- They have simplified the complexities of the game, making it tractable for game theoretic analysis.

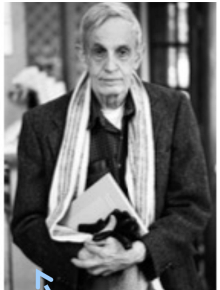


**Emile Borel**
1871 – 1956

**John von Neumann**
1903 – 1957

**Harold W. Kuhn**
1925 – 2014

**John Nash**
1928 – 2015



**Lloyd Shapley**
1923 –2016

- First to define the notion of games of strategy.
- Contributions to *Measure Theory* and *Probability* laid a robust foundation for modern mathematical analysis.
- Published several papers on poker, incorporating themes of imperfect information and credibility.
- Suggested the existence of *mixed strategies*— probability distributions over actions that can lead to equilibrium.

- Worked in the area of set theory, game theory, economic behavior, operator algebra, quantum mechanics, computer science, neural network, and the theory of automata.
- His great achievement in game theory was the book written with the Austrian economist O. Morgenstern.
- Co-invented the Monte Carlo Method with Stanislaw Ulam during World War II.
- The methods of Monte-Carlo and the duality theorem in LP are the two most distinguished results that he contributed in computer-oriented numerical analysis.

- An important figure in math programming and game theory.
- Developed Kuhn poker, a simplified version with three cards: King, Queen, and Jack
- Co-developed the *Kuhn-Tucker theorem and conditions* with Albert Tucker, fundamental in optimization.
- Developed Hungarian Method, an algorithm for the problem of assigning of workers to tasks. It was later shown to be the first algorithm of polynomial complexity for a large class of linear programs.

- John Nash & Lloyd Shapley's important paper: "A Simple Three-person Poker Game" (1950)

- Nash is known for the Nash Equilibrium— no player can benefit from changing their strategy unilaterally.
- Nash portrayed in the film *A Beautiful Mind*.
- Nash received the Nobel Prize in Economics in 1994.

- Shapley made fundamental contributions to the analysis of both cooperative and non-cooperative games.
- Some of his foundational ideas have led to the study of matching markets and to the thriving branch of practical economics known as 'market design'.
- Shapley received the Nobel Prize in Economics in 2012.

## Outline

- Summary of von Neumann Poker (1938): 2-player, continuous
- Game theory refresher
- von Neumann Poker: 2-player, discrete
- von Neumann Poker: 3-player, discrete
- von Neumann Poker: 3-player, continuous

## von Neumann Poker

- In 1938, John von Neumann proposed his now-famous mathematical model of poker, a game with an *uncountably infinite* deck.
- Player I and Player II are dealt (uniformly at random) two "cards", **real numbers** $x$**,** $y \in [0, 1]$.
- They each see their own card, but have no clue about the opponent's card.
- At the start they each **put \$1 into the pot** (the so called *ante*).
- Player I has the option to check or bet $\$b$, while Player II can only call or fold.

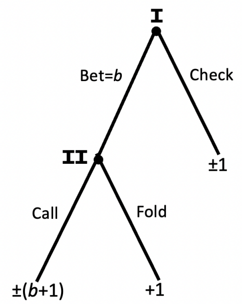
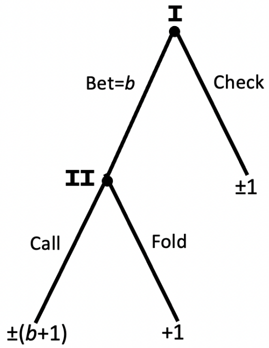
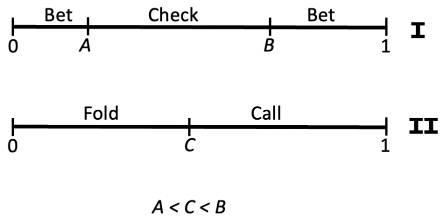



Fig: Betting tree

## von Neumann's pure Nash Equilibrium

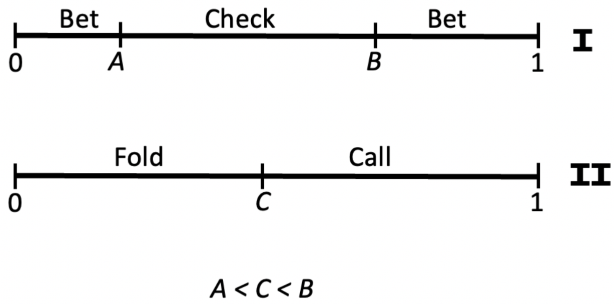



von Neumann proved that the following pair of strategies is a pure *Nash Equilibrium (NE)*, i.e. if the players both follow their chosen strategy, neither of them can do better (on average) by doing a different strategy.

- **Player I:** If $0 < x < \frac{b}{(b+4)(b+1)}$ or $\frac{b^2+4b+2}{(b+4)(b+1)} < x < 1$ you should **bet**, else **check**.
- **Player II:** If $0 < y < \frac{b(b+3)}{(b+4)(b+1)}$ you should **fold**, otherwise **call**.

The game favors Player I, and his expected gain is $\frac{b}{(b+4)(b+1)}$.



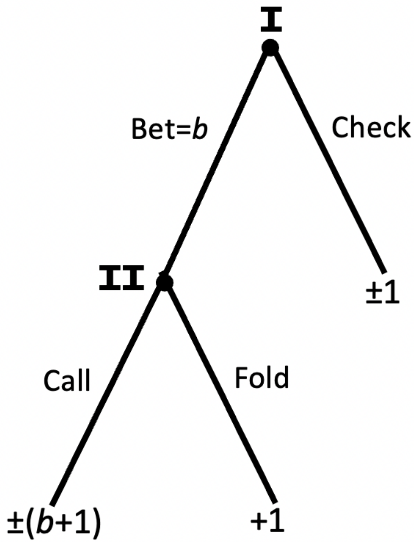

When $b = 2$, the advice spells out as follows:

- **Player I:** if $0 < x < \frac{1}{9}$ or $\frac{7}{9} < x < 1$ you should **bet**, otherwise **check**.
- **Player II:** If $0 < y < \frac{5}{9}$ you should **fold**, otherwise **call**.
- The expected value, i.e. the value of the game (for Player I) is $\frac{1}{9}$.

It can be shown that $b = 2$ maximizes Player I's payoff under the NE strategies.

# Finitely Many Cards

- In **real life** there are only *finitely* many cards, and in fact, not that many.
- We were wondering whether there exists pure Nash equilibria when there are only finitely many cards **1, 2, . . . , n**.



## Game Theory Refresher

**Payoff Matrix:** A table that describes the payoffs for each player based on the strategies chosen by both players in a game.

| | Player II plays 1 | Player II plays 2 | Player II plays 3 |
|---|---|---|---|
| Player I plays 1 | (8, 2) | (0, 9) | (7, 3) |
| Player I plays 2 | (3, 6) | (9, 0) | (2, 7) |
| Player I plays 3 | (1, 7) | (6, 4) | (8, 1) |
| Player I plays 4 | (4, 2) | (4, 6) | (5, 1) |

**Pure Nash Equilibrium:** A situation in a game where no player can benefit by changing their pure strategy while the other players keep theirs unchanged.

- given Player II's strategy, Player I is playing the best strategy he can (to maximize his payoff), and
- given Player I's strategy, Player II is playing the best strategy she can.

This concept is important because this strategy pair can be considered **stable** as neither player has an incentive to deviate from his choice.

| | Player II plays 1 | Player II plays 2 | Player II plays 3 |
|---|---|---|---|
| Player I plays 1 | (8, 2) | (0, 9) | (7, 3) |
| Player I plays 2 | (3, 6) | (9, 0) | (2, 7) |
| Player I plays 3 | (1, 7) | (6, 4) | (8, 1) |
| Player I plays 4 | (4, 2) | (4, 6) | (5, 1) |

**Play-safe strategy:** Each player looks for the worst that could happen if he makes each choice in turn. He then picks the choice that results in the least worst option.

- Player I calculates the minimum value for him each row. Then, select the maximum of these minimums. max (min ...)

- Player II calculates the minimum value for her each column. Then, select the maximum of these minimums.

* Both players will get better payoffs if they collaborate. e.g. (6,4), (7,3).

The **zero-sum game** is the game where the entries in each cell add up to 0. Collaboration does not give any advantageous in a zero-sum game (while it does in the non-zero sum game).

| | Player II plays 1 | Player II plays 2 | Player II plays 3 |
|---|---|---|---|
| Player I plays 1 | (3, -3) | (-4, 4) | (2, -2) |
| Player I plays 2 | (-1, 1) | (4, -4) | (-2, 2) |
| Player I plays 3 | (-3, 3) | (1, -1) | (4, -4) |
| Player I plays 4 | (1, -1) | (-1, 1) | (1, -1) |

A pay-off matrix of the zero-sum game is written from Player 1's point of view only:

$$\begin{vmatrix} 3 & -4 & 2 \\ -1 & 4 & -2 \\ -3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix}$$

**Important!** The pay-off of Player II in each entry is the negative of the entry.

## Play-safe strategies for the zero-sum game

- For Player I (row): the row maximin.
- For Player II (column): the column minimax. *(Player II aims to minimize their expected loss, or equivalently the expected gain of Player I.)*

The play safe strategy is
Player I plays 4,
player II plays 1

Is it NE ?

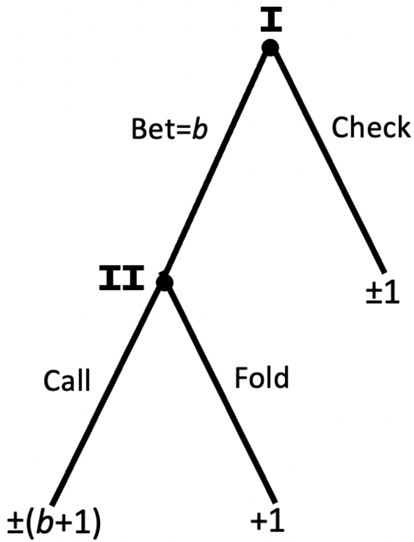
$$\begin{array}{r|rrr|c}
 & \text{min} \downarrow & & & \text{Row minimum} \\
 & 3 & -4 & 2 & -4 \\
 & -1 & 4 & -2 & -2 \\
 & -3 & 1 & 4 & -3 \\
 & \boxed{1} & -1 & 1 & -1 \leftarrow \text{max} \\
\hline
\text{Column maximum} & 3 & 4 & 4 &
\end{array}$$

**Theorem 1.** *In a zero-sum game there will be a pure NE if and only if*

$$\textit{the row maximin} = \textit{the column minimax.}$$

## Finitely Many Cards

- In **real life** there are only *finitely* many cards, and in fact, not that many.
- We were wondering whether there exists pure Nash equilibria when there are only finitely many cards **1, 2, . . . , n**.



## ➻ Finding all pure Nash Equilibria via the Maximin "Vanilla" approach

**Question:** How can we construct a payoff matrix with $n$ cards?

- A strategy for Player I can be *any* subset, $S_1$, of $\{1, \ldots, n\}$, that advises: 'If your card belongs to $S_1$ you should **bet**, otherwise, **check**'.
- Similarly a strategy for Player II, $S_2$, can be any such subset, that tells her to '**call** if her card $j \in S_2$, otherwise **fold**'.
- Thus, the payoff matrix can be obtained by listing outcomes of all pairs $[S_1, S_2]$

  - **Question:** What is the size of this payoff matrix?
- Once constructed, we look for pure NEs in the usual way:

*"If the row maximin equals the column minimax, then NEs exist."*

# Example: Payoff Matrix for $n = 2$ cards, Best size $b = 2$

Player II is the Column Player. She can either Call of Fold.

Paytable for 2 cards: {1,2}, and with bet size b=2.

Player I is the Row Player.

He can either Bet or Check.

| Strategy | $S_2 = \{\ \}$ Always Fold | $S_2 = \{1\}$ Call if "1", Fold if "2" | $S_2 = \{2\}$ Call if "2", Fold if "1" | $S_2 = \{1,2\}$ Always Call | Row Min |
|---|---|---|---|---|---|
| $S_1 = \{\ \}$ Always Check | (-1+1)/2 = 0 | (-1+1)/2 = 0 | (-1+1)/2 = 0 | (-1+1)/2 = 0 | |
| $S_1 = \{1\}$ Bet if "1", Check if "2" | (+1+1)/2 = 1 | (+1+1)/2 = 1 | (-3+1)/2 = -1 | (-3+1)/2 = -1 | |
| $S_1 = \{2\}$ Bet if "2", Check if "1" | (-1+1)/2 = 0 | (-1+3)/2 = 1 | (-1+1)/2 = 0 | (-1+3)/2 = 1 | |
| $S_1 = \{1,2\}$ Always Bet | 1 | (+1+3)/2 = 2 | (-3+1)/2 = -1 | (-3+3)/2 = 0 | |
| Column Max | | | | | |

## Example: Payoff Matrix for $n = 2$ cards, Best size $b = 2$

Player II is the Column Player. She can either Call of Fold.

Player I is the Row Player.

He can either Bet or Check.

| Strategy | $S_2 = \{\ \}$ Always Fold | $S_2 = \{1\}$ Call if "1", Fold if "2" | $S_2 = \{2\}$ Call if "2", Fold if "1" | $S_2 = \{1,2\}$ Always Call | Row Min |
|---|---|---|---|---|---|
| $S_1 = \{\ \}$ Always Check | (-1+1)/2 = 0 | (-1+1)/2 = 0 | (-1+1)/2 = 0 | (-1+1)/2 = 0 | 0 |
| $S_1 = \{1\}$ Bet if "1", Check if "2" | (+1+1)/2 = 1 | (+1+1)/2 = 1 | (-3+1)/2 = -1 | (-3+1)/2 = -1 | -1 |
| $S_1 = \{2\}$ Bet if "2", Check if "1" | (-1+1)/2 = 0 | (-1+3)/2 = 1 | (-1+1)/2 = 0 | (-1+3)/2 = 1 | 0 |
| $S_1 = \{1,2\}$ Always Bet | 1 | (+1+3)/2 = 2 | (-3+1)/2 = -1 | (-3+3)/2 = 0 | -1 |
| Column Max | 1 | 2 | 0 | 1 | |

Paytable for 2 cards: {1,2}, and with bet size b=2.

Row maximin = Column minimax = 0

So there are TWO pure NEs.

In both of them,

- Player II calls if her card is "2" and folds if her card is "1", while

- Player I always checks in the first strategy, and checks if his card is "1" in the second strategy.

This is not very interesting, since the expected gain (value of the the game) is 0.

**Let's fix the bet size $b = 2$, and consider the pure NEs for other $n$ cards.**

- If the card has only 2 cards, `vnNE(2,2);` gives

$$[\phi, \{2\}] \text{ and } [\{2\}, \{2\}]$$

- `vnNE(3,2);` is equally boring, giving the two trivial pairs $[\phi, \{3\}]$ and $[\{3\}, \{3\}]$
- `vnNE(4,2);`, `vnNE(5,2;)`, and `vnNE(6,2);` are even more boring, they are empty! That is, there is no pure NEs.

```
> vnNE(2,2);
                {[{ }, {2}, 0], [{2}, {2}, 0]}
> vnNE(3,2);
                {[{ }, {3}, 0], [{3}, {3}, 0]}
> vnNE(4,2);
                             { }
> vnNE(5,2);
                             { }
> vnNE(6,2);
                             { }
```

Maple package: `https://sites.math.rutgers.edu/~zeilberg/tokhniot/FinitePoker.txt`.

sites.math.rutgers.edu

**von Neumann and Newman Pokers with Finite Decks**

**By Tipaluck Krityakierne, Thotsaporn "Aek" Thanatipanonda, and Doron Zeilberger**

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**.pdf** **.tex**

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**First Written: July 22, 2024.**

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**John von Neumann and Donald J. Newman proposed, and brilliantly solved, toy models of poker where the cards are drawn from an infinite deck (in fact very infinite, the set of real numbers from 0 to 1). We show the power of symbolic computation by implementing, and experimenting with, these models for finite decks of cards.**

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## Pictures

**John Nash and friends**

**Also See pictures,**

## Maple packages

- **FinitePoker.txt, a Maple package for finding pure and mixed Nash Equilbria for poker with a finite number of cards, doing it completely *ab initio***
- **ThreePersonPoker.txt, a Maple package for studying three person poker in the footsteps of John Nash**

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## Sample Input and Output for FinitePoker.txt

- **If you want to see ALL pure Nash equilibria (and one mixed one) for von Neumann poker with number of cards from 2 to 10, and bet sizes from 1 to 5 and one mixed one the input gives the output.**
- **If you want to see ALL pure Nash equilibria, and one mixed one, for von Neumann poker with number of cards from 2 to 11, and bet sizes from 1 to 3 the input gives the output.**

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- **If you want to see ALL pure Nash equilibria and one mixed one, for von Neumann poker with number of cards from 2 to 27, and bet size 2 only using suggested strategies the input gives the output.**

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- **If you want to see ALL pure Nash equilibria and one mixed one, for DJ Newman poker with number of cards from only using suggested strategies the input gives the output.**

# Payoff Matrix for $n = 4$ cards, Best size $b = 2$

| Strategy | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | Row Min |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 ← |
| 2 | 1/2 | 1/2 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | -1/6 | -1/6 | -1/6 | -1/6 | -1/6 | -1/6 | -1/2 | -1/2 | |
| 3 | 1/3 | 1/2 | 1/3 | 0 | 0 | 1/2 | 1/6 | 1/6 | 0 | 0 | -1/3 | 1/6 | 1/6 | -1/6 | -1/3 | -1/6 | |
| 4 | 1/6 | 1/3 | 1/3 | 1/6 | -1/6 | 1/2 | 1/3 | 0 | 1/3 | 0 | -1/6 | 1/2 | 1/6 | 0 | 0 | 1/6 | |
| 5 | 0 | 1/6 | 1/6 | 1/6 | 0 | 1/3 | 1/3 | 1/6 | 1/3 | 1/6 | 1/6 | 1/2 | 1/3 | 1/3 | 1/3 | 1/2 | 0 ← |
| 6 | 5/6 | 1 | 1/2 | 1/6 | 1/6 | 2/3 | 1/3 | 1/3 | -1/6 | -1/6 | -1/2 | 0 | 0 | -1/3 | -5/6 | -2/3 | |
| 7 | 2/3 | 5/6 | 1/2 | 1/3 | 0 | 2/3 | 1/2 | 1/6 | 1/6 | -1/6 | -1/3 | 1/3 | 0 | -1/6 | -1/2 | -1/3 | |
| 8 | 1/2 | 2/3 | 1/3 | 1/3 | 1/6 | 1/2 | 1/2 | 1/3 | 1/6 | 0 | 0 | 1/3 | 1/6 | 1/6 | -1/6 | 0 | |
| 9 | 1/2 | 5/6 | 2/3 | 1/6 | -1/6 | 1 | 1/2 | 1/6 | 1/3 | 0 | -1/2 | 2/3 | 1/3 | -1/6 | -1/3 | 0 | |
| 10 | 1/3 | 2/3 | 1/2 | 1/6 | 0 | 5/6 | 1/2 | 1/3 | 1/3 | 1/6 | -1/6 | 2/3 | 1/2 | 1/6 | 0 | 1/3 | |
| 11 | 1/6 | 1/2 | 1/2 | 1/3 | -1/6 | 5/6 | 2/3 | 1/6 | 2/3 | 1/6 | 0 | 1 | 1/2 | 1/3 | 1/3 | 2/3 | |
| 12 | 1 | 4/3 | 5/6 | 1/3 | 0 | 7/6 | 2/3 | 1/3 | 1/6 | -1/6 | -2/3 | 1/2 | 1/6 | -1/3 | -5/6 | -1/2 | |
| 13 | 5/6 | 7/6 | 2/3 | 1/3 | 1/6 | 1 | 2/3 | 1/2 | 1/6 | 0 | -1/3 | 1/2 | 1/3 | 0 | -1/2 | -1/6 | |
| 14 | 2/3 | 1 | 2/3 | 1/2 | 0 | 1 | 5/6 | 1/3 | 1/2 | 0 | -1/6 | 5/6 | 1/3 | 1/6 | -1/6 | 1/6 | |
| 15 | 1/2 | 1 | 5/6 | 1/3 | -1/6 | 4/3 | 5/6 | 1/3 | 2/3 | 1/6 | -1/3 | 7/6 | 2/3 | 1/6 | 0 | 1/2 | |
| 16 | 1 | 3/2 | 1 | 1/2 | 0 | 3/2 | 1 | 1/2 | 1/2 | 0 | -1/2 | 1 | 1/2 | 0 | -1/2 | 0 | |
| Column Max | | | | | 1/6 | | | | | 1/6 | 1/6 | | | | | | |

Paytable for 4 cards: {1,2,3,4}, and with bet size b=2.

| Strategy | Player I bets if... / Player II calls if... |
|---|---|
| 1 | {} |
| 2 | {1} |
| 3 | {2} |
| 4 | {3} |
| 5 | {4} |
| 6 | {1, 2} |
| 7 | {1, 3} |
| 8 | {1, 4} |
| 9 | {2, 3} |
| 10 | {2, 4} |
| 11 | {3, 4} |
| 12 | {1, 2, 3} |
| 13 | {1, 2, 4} |
| 14 | {1, 3, 4} |
| 15 | {2, 3, 4} |
| 16 | {1, 2, 3, 4} |

Row maximin = 0

Column minimax = 1/6

≠ No Pure NE!

But now comes a nice surprise, `vnNE(7,2);` gives three pure, *non-trivial*, NEs.

- For all of them Player I bets if his card belongs to $\{1,6,7\}$. Player II calls if her card is in $\{3,6,7\}$, $\{4,6,7\}$, or $\{5,6,7\}$. The value of the game is $\frac{2}{21}$.
- So with 7 cards we already have bluffing! If Player I has the card labeled 1, he should bet even though he would definitely lose the bet if Player II calls.

```
> vnNE(7,2);
```

$$\left\{\left[\{1,6,7\},\{3,6,7\},\frac{2}{21}\right],\left[\{1,6,7\},\{4,6,7\},\frac{2}{21}\right],\left[\{1,6,7\},\{5,6,7\},\frac{2}{21}\right]\right\}$$

Moving right along, `vnNE(8,2);` also gives you three pure NEs.

- For all of them Player I bets if his card belongs to $\{1,7,8\}$, but Player II calls if her card is in either $\{4,7,8\}$, $\{5,7,8\}$, or $\{6,7,8\}$. The value of the game is $\frac{3}{28}$, getting tantalizingly close to von Neumann's $\frac{1}{9}$.

```
> vnNE(8,2);
```

$$\left\{\left[\{1,7,8\},\{4,7,8\},\frac{3}{28}\right],\left[\{1,7,8\},\{5,7,8\},\frac{3}{28}\right],\left[\{1,7,8\},\{6,7,8\},\frac{3}{28}\right]\right\}$$

## ➽ Curse of dimensionality!

Since the size $2^n$ by $2^n$ of the payoff matrix grows exponentially, and we did not make *any plausibility assumptions*, there is only so far we can go with this naive ***vanilla*** approach.

But nine cards, with $512 \times 512$, paytable are still doable.

Indeed, `vnNE(9,2);` gives you seven pure NEs in this case.

- For all of them $S_1 = \{1, 8, 9\}$, but Player II has seven choices, all with four members, including, of course, $\{6, 7, 8, 9\}$.

```
> vnNE(9,2);
```

$$\{[\{1,8,9\},\{3,6,8,9\},\tfrac{1}{9}],[\{1,8,9\},\{3,7,8,9\},\tfrac{1}{9}],[\{1,8,9\},\{4,6,8,9\},\tfrac{1}{9}],[\{1,8,9\},\{4,7,8,9\},\tfrac{1}{9}],$$
$$[\{1,8,9\},\{5,6,8,9\},\tfrac{1}{9}],[\{1,8,9\},\{5,7,8,9\},\tfrac{1}{9}],[\{1,8,9\},\{6,7,8,9\},\tfrac{1}{9}]\}$$

## ➽ Mixed NEs

**Mixed Strategy:** A strategy where a player randomizes over two or more pure strategies, assigning a probability to each option.

**Expected Payoff:** The anticipated value of a player's payoff, calculated as the sum of possible payoffs, each weighted by its probability of occurrence.

**Nash Equilibrium:** A situation in a game where no player can benefit by changing their strategy while the other players keep theirs unchanged.

**von Neumann's Theorem (1928):** Every finite two-person zero-sum game has at least one Nash equilibrium in mixed strategies. They are the maximin mixed strategies.

## ➽ Mixed NEs via Linear Programming

- The study of **mixed strategies** in two-player zero-sum games can be elegantly formulated as a primal-dual **linear programming (LP)** problem.

- A **mixed strategy** involves each player choosing optimal actions according to a probability distribution, introducing uncertainty.

- An equilibrium solution to this dual pair of linear programs reveals optimal mixed strategies (mixed NE) for both players.

- Given the $2^n$ by $2^n$ payoff matrix $(m_{ij})$ as input, Player I aims to maximize his worst-case expected gain, minimizing over all possible actions of Player II.

## Mixed strategies for $n = 2$ cards

$$\max_{\sum_{i=1}^{4} x_i = 1} \Big( \min \big( \ \times\ ,\ \ \times\ ,\ \ \times\ ,\ \ \times\ \big) \Big)$$

Player 1

$\sum_{j=1}^{4} y_j = 1$

$0 \leq y_j \leq 1$

$\sum_{i=1}^{4} x_i = 1,$

$0 \leq x_i \leq 1$

| | Player II plays with probabilities | y1 | y2 | y3 | y4 |
|---|---|---|---|---|---|
| **Player I plays with probabilities** | **Strategy** | $S_2 = \{\ \}$ Always Fold | $S_2 = \{1\}$ Call if "1", Fold if "2" | $S_2 = \{2\}$ Call if "2", Fold if "1" | $S_2 = \{1,2\}$ Always Call |
| x1 | $S_1 = \{\ \}$ Always Check | 0 | 0 | 0 | 0 |
| x2 | $S_1 = \{1\}$ Bet if "1", Check if "2" | 1 | 1 | -1 | -1 |
| x3 | $S_1 = \{2\}$ Bet if "2", Check if "1" | 0 | 1 | 0 | 1 |
| x4 | $S_1 = \{1,2\}$ Always Bet | 1 | 2 | -1 | 0 |

## Mixed strategies for $n = 2$ cards

- Let $\mathbf{x} = (x_1, x_2, x_3, x_4)$ be the mixed strategy probability of Player I.
- Let $\mathbf{y} = (y_1, y_2, y_3, y_4)$ be the mixed strategy probability of Player II.

| | | y1 | y2 | y3 | y4 |
|---|---|---|---|---|---|
| | Strategy | 1 | 2 | 3 | 4 |
| x1 | 1 | $m_{11}$ | $m_{12}$ | $m_{13}$ | $m_{14}$ |
| x2 | 2 | $m_{21}$ | $m_{22}$ | $m_{23}$ | $m_{24}$ |
| x3 | 3 | $m_{31}$ | $m_{32}$ | $m_{33}$ | $m_{34}$ |
| x4 | 4 | $m_{41}$ | $m_{42}$ | $m_{43}$ | $m_{44}$ |

Player I (Primal problem=Maximin the Payoff):

$$\max_{\substack{x_1+x_2+x_3+x_4=1 \\ 0 \le x_i \le 1}} \left\{ \min \left( \sum_{i=1}^{4} x_i m_{i1}, \sum_{i=1}^{4} x_i m_{i2}, \sum_{i=1}^{4} x_i m_{i3}, \sum_{i=1}^{4} x_i m_{i4} \right) \right\}$$

Player II (Dual problem=Minimax the Loss):

$$\min_{\substack{y_1+y_2+y_3+y_4=1 \\ 0 \le y_j \le 1}} \left\{ \max \left( \sum_{j=1}^{4} m_{1j} y_j, \sum_{j=1}^{4} m_{2j} y_j, \sum_{j=1}^{4} m_{3j} y_j, \sum_{j=1}^{4} m_{4j} y_j \right) \right\}$$

## ➽ Slow LPs for mixed NE

Player I (Primal problem):

$$\max_{\substack{x_1+x_2+x_3+x_4=1\\ 0\le x_i\le 1}} \left\{ \min\left( \sum_{i=1}^{4} x_i m_{i1}, \sum_{i=1}^{4} x_i m_{i2}, \sum_{i=1}^{4} x_i m_{i3}, \sum_{i=1}^{4} x_i m_{i4} \right) \right\}$$

Primal: Maximize $v_1$

$$\text{s.t.} \sum_{i=1}^{2^n} x_i \cdot m_{ij} \ge v_1 \quad \text{for } j = 1, ..., 2^n$$

$$\sum_{i=1}^{2^n} x_i = 1$$

$$x_i \ge 0 \quad \text{for } i = 1, ..., 2^n.$$

Player II (Dual problem):

$$\min_{\substack{y_1+y_2+y_3+y_4=1\\ 0\le y_j\le 1}} \left\{ \max\left( \sum_{j=1}^{4} m_{1j} y_j, \sum_{j=1}^{4} m_{2j} y_j, \sum_{j=1}^{4} m_{3j} y_j \sum_{j=1}^{4} m_{4j} y_j \right) \right\}$$

Dual: Minimize $v_2$

$$\text{s.t.} \sum_{j=1}^{2^n} m_{ij} \cdot y_j \le v_2 \quad \text{for } i = 1, ..., 2^n$$

$$\sum_{j=1}^{2^n} y_j = 1$$

$$y_j \ge 0 \quad \text{for } j = 1, ..., 2^n.$$

By the **minimax theorem** at an equilibrium, $v_1 = v_2 = v^*$, which represents **the value of the game**.

We have implemented the above LP procedure in `MNE(M,S1,S2)`.

For example, when $n = 4$ cards and $b = 2$, type:

```
M := PayTable(4,2)[1]:
S1 := Stra1(4):
S2 := Stra2(4):
MNE(M,S1,S2);
```

gives outputs:

`[ {[{4}, 1/2], [{1,4}, 1/2]}, {[{4}, 1/2], [{2,4}, 1/2]}, ` **1/12** ` ]`

Translation:

- The value of the game is **1/12** (last entry).
- Player I has two strategies specified within the first set of braces:
  - with probability 1/2, bet if his card is 4 and check if his cards are 1, 2, or 3;
  - with probability 1/2, bet if his card is 1 or 4 and check if his cards are 2 or 3.
- Player II has two strategies specified within the second set of braces:
  - with probability 1/2, call if her card is 4 and fold if her cards are 1, 2, or 3;
  - with probability 1/2, call if her cards are 2 or 4 and fold if her cards are 1 or 3.

We call the above approach "Slow LP" procedure because it is really **SLOW!**

Due to the **exponentially large** size of the matrix, it restricts us from considering more than 6-7 cards without the inconvenience of reducing the dominated rows/columns of the payoff matrix.

This is worse than the vanilla approach. ☹

**Question:** Is it possible to reduce the number of constraints?

## ➽ Fast LPs for mixed NE

The answer is **Yes!**

**Trick:** Focus on the “card-by-card” strategies rather than the “all-cards” strategies.

With this formulation, we can reduce the number of constraints **from exponential to linear**.

*Let’s explore how this works...*

## Card-by-card strategies

- A strategy for Player I is given by a vector $\mathcal{P} = [p_1, \ldots, p_n]$ that tells him:
  - if his card is $i$, bet with probability $p_i$,
  - and check with probability $1 - p_i$.
- A strategy for Player II is given by a vector $\mathcal{Q} = [q_1, \ldots, q_n]$ that tells her:
  - if her card is $j$, call with probability $q_j$,
  - and fold with probability $1 - q_j$.

## ➽ Fast LPs for mixed NE of Player I

Each sets of constraints **Player I** corresponds to the expected payoff (over distribution $\mathcal{P}$), conditioned on the card that Player II has and whether she calls or folds:

$$
\begin{aligned}
&\text{Maximize } \frac{1}{n}\sum_{j=1}^{n} v_j \\
\text{s.t. } &\frac{1}{n-1}\sum_{i\neq j}(\texttt{Call}(i,j,b+1)\cdot p_i + \texttt{Call}(i,j,1)\cdot(1-p_i)) \geq v_j \quad j=1,\ldots,n \text{ (Player II calls)} \\
&\frac{1}{n-1}\sum_{i\neq j}(p_i + \texttt{Call}(i,j,1)\cdot(1-p_i)) \geq v_j \quad j=1,\ldots,n \text{ (Player II folds)} \\
&0 \leq p_i \leq 1 \quad i=1,\ldots,n, \qquad\qquad \text{(VN-I)}
\end{aligned}
$$

---

where

$$
\texttt{Call}(i,j,R) = \begin{cases} R & \text{if } i > j \\ -R & \text{if } i < j. \end{cases}
$$

## ➽ Fast LPs for mixed NE Player II

Similarly, for the Fast LP for **Player II**, the constraints are calculated based on the expected loss (over distribution $\mathcal{Q}$), conditioned on the card that Player I has and whether he raises or checks:

$$
\begin{aligned}
&\text{Minimize } \frac{1}{n}\sum_{i=1}^{n} v_i \\
\text{s.t. } &\frac{1}{n-1}\sum_{j\neq i} (\texttt{Call}(i,j,b+1)\cdot q_j + (1-q_j)) \leq v_i \quad i = 1,\ldots,n \text{ (Player I raises)} \\
&\frac{1}{n-1}\sum_{j\neq i} \texttt{Call}(i,j,1) \leq v_i \quad i = 1,\ldots,n \text{ (Player I checks)} \\
&0 \leq q_j \leq 1 \quad j = 1,\ldots,n.
\end{aligned}
\tag{VN-II}
$$

With 3 cards and bet size 1, typing `lprint(vnMNE(3,1));` outputs:

```
[ 1/18, .5555555556e-1, [1/3, 0, 1], [0, 1/3, 1] ]   .
```

Translation:

- The value of the game is **1/18**.
- Its value in decimals is $0.055555\ldots$.
- Player I's strategy is: If your card is 1, bet with probability $\frac{1}{3}$ and check with probability $\frac{2}{3}$. If your card is 2 then **definitely check**, while if your card is 3 then you should **definitely bet**.
- Player II's strategy is: If your card is 1, **definitely fold**, if your card is 2, call with probability $\frac{1}{3}$ and fold with probability $\frac{2}{3}$, while if your card is 3 then **definitely call**.

So already with three cards, Player I should sometimes bluff if his card is 1, but only with probability $\frac{1}{3}$.

Note that a pure NE is also a mixed one, and indeed sometimes we get pure NEs. For example,

`lprint(vnMNE(9,2));` gives:

`[` **1/9**`, .1111111111, [1, 0, 0, 0, 0, 0, 0, 1, 1], [0, 0, 0, 0, 0, 1, 1, 1, 1 ] ].`

Translation:

- The value of the game is $\mathbf{1/9}$.
- The value of the game in floating-point is $0.111111111\ldots$.
- Player I: Bet iff your card is in $\{1, 8, 9\}$.
- Player II: Call iff your card is in $\{6, 7, 8, 9\}$.

- This was *so much faster* than the "vanilla" approach for pure NEs.
- With this fast LP formulation, we can handle more than 200 cards now.

## ➽ Three-player Poker Game

- As early as 1950, future Economics Nobelists, John Nash and Lloyd Shapley, pioneered the analysis of a three-player poker game.
- They explored a simplified version where the deck contains only two kinds of cards, High and Low, in equal numbers.
- Today, eighty years after von Neumann's 1938 analysis of poker, the dynamics of the three-player game therein remain unexplored.
- We now take the opportunity to analyze these dynamics in both their finite and infinite versions.

## ➽ Three players, Finite deck

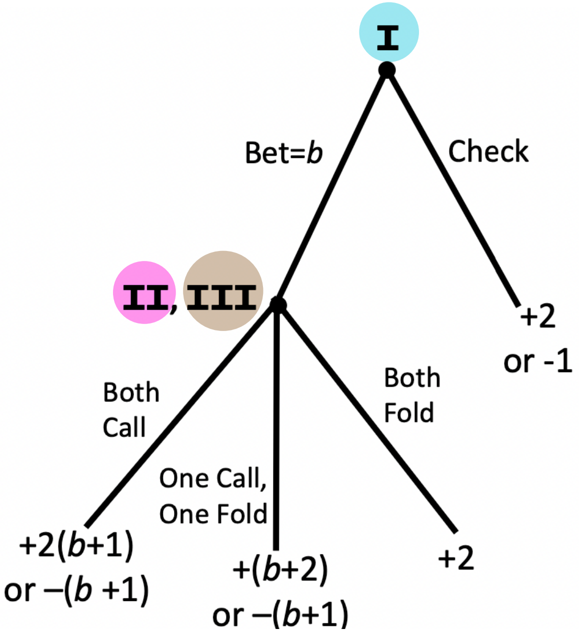



Fig: Betting tree

## ➽ Three players, Finite deck

- While the two-player game can be solved using LP, here **we require NLP**.
- The NLP formulation for the three-player game closely follows the LP model for the two players.
- **Each player aims to minimize their expected loss, or the expected gain of the other players.**

  Assume we are given 3-D payoff matrices $\left(M^l, l = 1, 2, 3\right)$ for the three players:

$$M_l = \left(m^l_{ijk}\right), \qquad \text{where } i, j, k = 1, 2, \ldots, 2^n.$$

- E.g, given Player I's payoff matrix $M^1$, Players II and III attempt to **minimize the maximum potential loss incurred due to Player I's choices**.
  - This involves constraints that utilize matrix $M^1$ and the probability distributions $\mathbf{y} = (y_1, \ldots, y_{2^n})$ and $\mathbf{z} = (z_1, \ldots, z_{2^n})$ of Players II and III.
  - These are embedded in the first set of constraints in the NLP formulation, which we will now formulate.

## ➽ Slow NLP for three players

$$
\begin{aligned}
&\text{Minimize } \sum_{l=1}^{3} v^l \\
\text{s.t. } &\sum_{j,k=1}^{2^n} m^1_{ijk} \cdot y_j \cdot z_k \le v^1 \quad \text{for } i = 1, 2, ..., 2^n \\
&\sum_{i,k=1}^{2^n} m^2_{ijk} \cdot x_i \cdot z_k \le v^2 \quad \text{for } j = 1, 2, ..., 2^n \\
&\sum_{i,j=1}^{2^n} m^3_{ijk} \cdot x_i \cdot y_j \le v^3 \quad \text{for } k = 1, 2, ..., 2^n \\
&\sum_{i=1}^{2^n} x_i = 1, \qquad \sum_{j=1}^{2^n} y_j = 1, \qquad \sum_{k=1}^{2^n} z_k = 1 \\
&x_i, y_j, z_k \ge 0 \quad \text{for } i, j, k = 1, 2, ..., 2^n.
\end{aligned}
$$

Yet, this is **SLOW**!

**Remark: NLP of three players can be reduced to LP of two players**

Three players (NLP)

$$\text{Minimize} \sum_{l=1}^{3} v^l \longrightarrow \text{Minimize } v^1 + v^2$$

$$\text{s.t.} \sum_{j,k=1}^{2^n} m^1_{ij\cancel{k}} \cdot y_j \cdot \cancel{z_k} \le v^1 \quad \text{for } i = 1, 2, \ldots, 2^n$$

$$\sum_{i,k=1}^{2^n} m^2_{ij\cancel{k}} \cdot x_i \cdot \cancel{z_k} \le v^2 \quad \text{for } j = 1, 2, \ldots, 2^n$$

$$\cancel{\sum_{i,j=1}^{2^n} m^3_{ijk} \cdot x_i \cdot y_j \le v^3 \quad \text{for } k = 1, 2, \ldots, 2^n}$$

$$\sum_{i=1}^{2^n} x_i = 1, \qquad \sum_{j=1}^{2^n} y_j = 1, \qquad \cancel{\sum_{k=1}^{2^n} z_k = 1}$$

$$x_i, y_j, z_k \ge 0 \quad \text{for } i, j, k = 1, 2, \ldots, 2^n.$$

Two players
(Player II's LP)

Dual: Minimize $v_2$

$$\text{s.t.} \sum_{j=1}^{2^n} m_{ij} \cdot y_j \le v_2 \quad \text{for } i = 1, \ldots, 2^n$$

$$\sum_{j=1}^{2^n} y_j = 1$$

$$y_j \ge 0 \quad \text{for } j = 1, \ldots, 2^n.$$

## ➽ Fast NLP for three players: Card-by-card strategy

- Strategy for Player I is given by a vector $\mathcal{P} = [p_1, \ldots, p_n]$, indicating that if his card is $i$, he bets with probability $p_i$, and checks with probability $1 - p_i$.
- Strategy for Player II is given by a vector $\mathcal{Q} = [q_1, \ldots, q_n]$, indicating that if her card is $j$, she calls with probability $q_j$, and folds with probability $1 - q_j$.
- Similarly, a strategy for Player III is represented by a vector $\mathcal{R} = [r_1, \ldots, r_n]$, following the same interpretation as Player II.

## ➽ Fast NLP for three players: Objective function

$$\text{Minimize } \frac{1}{n}\sum_{c=1}^{n} v_c^1 + \frac{1}{n}\sum_{c=1}^{n} v_c^2 + \frac{1}{n}\sum_{c=1}^{n} v_c^2$$

**Overview of Constraints:**

- The Fast NLP contains three sets of constraints—one set for each player—corresponding to minimizing the expected loss of the other two players over the pairs of distributions $\mathcal{Q} - \mathcal{R}$, $\mathcal{P} - \mathcal{R}$, or $\mathcal{P} - \mathcal{Q}$.
- For each player $l = 1, 2, 3$, there are two sets of constraints depending on the card that Player $l$ has and whether they follow their first strategy or the second strategy

## ➽ Fast NLP for three players

**Two sub-procedures:**

- `Call2` is used to calculate the payoff if **either** Player II or Player III decides to **fold**, leaving only two players (one of whom is Player I) to compare their cards. Let us assume that Player III folds. Then,

$$\texttt{Call2}(i, j, R) = \begin{cases} R + 1 & \text{if } i > j \\ -R & \text{if } i < j. \end{cases}$$

- `Call3` is used to calculate the payoff when **all the three players are comparing** their cards:

$$\texttt{Call3}(i, j, k, R) = \begin{cases} 2R & \text{if } i > j \text{ and } i > k \\ -R & \text{if } i < j \text{ or } i < k. \end{cases}$$

## ➠ First set of constraints due to Player I

Players II and II try to minimize their expected loss due to Player I.

$$\frac{1}{(n-1)(n-2)} \sum_{j \neq i} \sum_{k \neq i,j} \texttt{Call3}(i,j,k,1) \leq v_i^1 \quad i = 1,\ldots,n \qquad \text{(Player I checks)}$$

$$\frac{1}{(n-1)(n-2)} \Bigg( \sum_{j \neq i} \sum_{k \neq i,j} \texttt{Call3}(i,j,k,b+1) \cdot q_k \cdot r_k$$
$$+ \texttt{Call2}(i,j,b+1) \cdot q_j \cdot (1-r_k) + \texttt{Call2}(i,k,b+1) \cdot (1-q_j) \cdot r_k$$
$$+ 2(1-q_j) \cdot (1-r_k) \Bigg) \leq v_i^1 \quad i = 1,\ldots,n \qquad \text{(Player I bets)}$$

## ➽ Second set of constraints due to Player II

Players I and III try to minimize their expected loss due to Player II.

$$\frac{1}{(n-1)(n-2)} \sum_{i \neq j} \sum_{k \neq i,j} (-p_i + \texttt{Call3}(j,i,k,1) \cdot (1-p_i)) \leq v_j^2 \quad j = 1, \ldots, n \qquad \text{(Player II folds)}$$

$$\frac{1}{(n-1)(n-2)} \Bigg( \sum_{i \neq j} \sum_{k \neq i,j} \texttt{Call3}(j,i,k,b+1) \cdot p_i \cdot r_k$$
$$+ \texttt{Call2}(j,i,b+1) \cdot p_i \cdot (1-r_k) + \texttt{Call3}(j,i,k,1) \cdot (1-p_i) \Bigg) \leq v_j^2 \quad j = 1, \ldots, n$$
$$\text{(Player II calls)}$$

## ➽ Third set of constraints due to Player III

Players I and II try to minimize their expected loss due to Player III.

$$\frac{1}{(n-1)(n-2)} \sum_{i \neq k} \sum_{j \neq i,k} (-p_i + \texttt{Call3}(k,i,j,1) \cdot (1-p_i)) \leq v_k^3 \quad k = 1, \ldots, n \qquad \text{(Player III folds)}$$

$$\frac{1}{(n-1)(n-2)} \Bigg( \sum_{i \neq k} \sum_{j \neq i,k} \texttt{Call3}(k,i,j,b+1) \cdot p_i \cdot q_j + \texttt{Call2}(k,i,b+1) \cdot p_i \cdot (1-q_j) + \texttt{Call3}(k,i,j,1) \cdot (1-p_i) \Bigg) \leq v_k^3 \quad k = 1, \ldots, n$$

(Player III calls)

$$0 \leq p_i, q_j, r_k \leq 1 \quad i, j, k = 1, \ldots, n.$$

Assume that Players II and III adopt identical strategies.

With 4 cards, and bet size 1,

```
FastMNE(4,1);
```
gives

```
[ [0, 1/24, -1/48, -1/48], [2/3, 0, 0, 1], [0, 0, 1/4, 1], [0, 0, 1/4, 1]
].
```

Translation:

- The value of the game (for Player 1) is **1/24**, while for Players II and III are -1/48 each.
- Player I's strategy is: If your card is 1, bet with probability of $\frac{2}{3}$ and check with probability $\frac{1}{3}$. If your card is 2 or 3, then definitely checks; if your card is 4, definitely bet.
- Player II's and Player III 's strategies are: If their card is 1 or 2, they definitely fold. If their card is 3, they call with probability of $\frac{1}{4}$ and fold with probability $\frac{3}{4}$. If their card is 4, they definitely call.

Another example, with 10 cards, and bet size 2,
`FastMNE(10,2);` produces

```
[ [0, 106/1125, -53/1125, -53/1125], [16/19, 0, 0, 0, 0, 0, 0, 0, 0, 1],
[0, 0, 0, 0, 0, 0, 3/25, 1, 1, 1], [0, 0, 0, 0, 0, 0, 3/25, 1, 1, 1] ].
```

Translation:

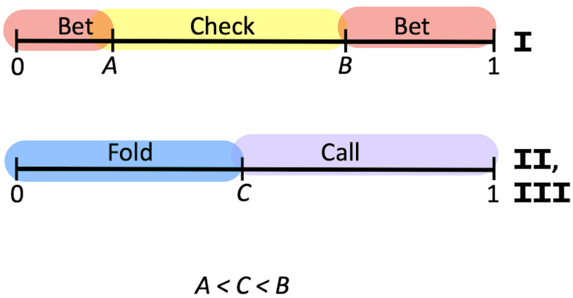
- The value of the game (for Player 1) is **106/1125**, while for Players II and III are -53/1125 each.
- Player I's strategy is: If your card is 1, bet with probability of $\frac{16}{19}$ and check with probability $\frac{3}{19}$. If your card is 2 to 9, then definitely checks; if your card is 10, definitely bet.
- Player II's and Player III 's strategies are: If their card is 1 to 6, they definitely fold. If their card is 7, they call with probability of $\frac{3}{25}$ and fold with probability $\frac{22}{25}$. If their card is 8 to 10, they definitely call.

Another example, with 24 cards, and bet size 2,
`FastMNE(24,2);` produces

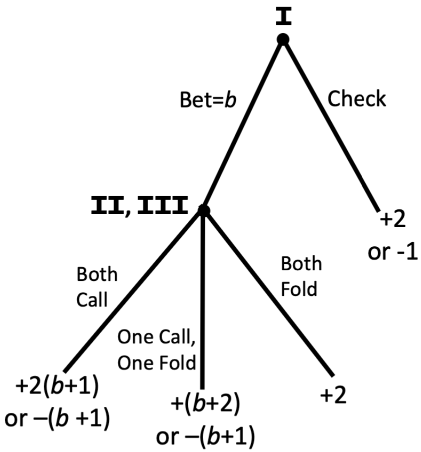
```
[ [0, 1106/9547, -553/9547, -553/9547], [1, 1, 1, 7/27, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 1/7, 1, 1, 1, 1, 1, 1, 1, 1], [0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 1/7, 1, 1, 1, 1, 1, 1, 1, 1] ] .
```

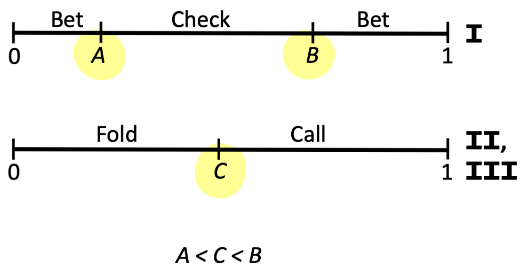
Translation?

Extension to a continuous model?



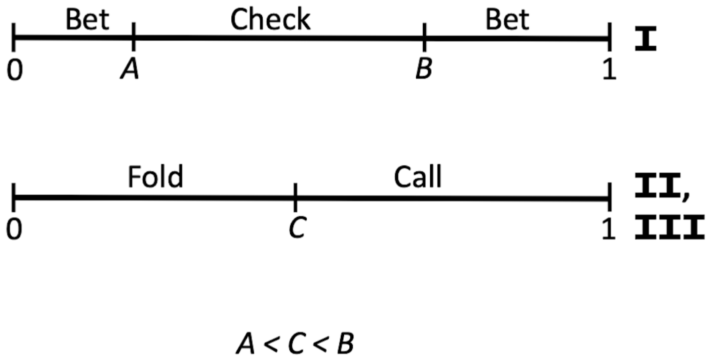
## ➽ Extension of von Neumann's continuous model to three players




- Each of the three players contributes 1 dollar to the pot and receives independent uniform(0,1) hands.
- Player I has the option to check or bet a fixed amount $b$, while Player II and Player III can only call or fold.

## Determining the NE strategies




For numbers $A, B, C$, yet *to be determined*,

- Player I: If $0 < x < A$ or $B < x < 1$ he should **bet**, otherwise **check**.
- Player II and III: If $0 < y < C$ they should **fold**, otherwise **call**.

**The Principle of Indifference** states that, in mixed strategy Nash equilibria, players are indifferent between their available strategies because each strategy yields the same expected payoff.

**Theorem 2** (The Equilibrium Theorem)**.** *Consider a two-player, zero sum game with $n_1 \times n_2$ payoff matrix $M$ and value of the game $v$. Let $\mathbf{x} = (x_1, \dots, x_{n_1})$ be a mixed strategy probability of Player I and $\mathbf{y} = (y_1, \dots, y_{n_2})$ be a mixed strategy probability of Player II. Then,*

$$\sum_{i=1}^{n_1} x_i m_{ij} = v \qquad \textit{for all } j \textit{ for which } y_j > 0.$$

*and*

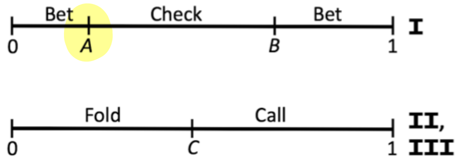
$$\sum_{j=1}^{n_2} m_{ij} y_j = v \qquad \textit{for all } i \textit{ for which } x_i > 0.$$

Theorem 2 is useful for helping **direct us toward the solution:**

- Player I searches for a strategy $\mathbf{x} = (x_1, \dots, x_{n_1})$ that makes Player II indifferent as to which of the (good) pure strategies to use.
- Player II should play in such a way (searching for $\mathbf{y} = (y_1, \dots, y_{n_2})$) to make Player I indifferent among his (good) strategies.

This is called the **Principle of Indifference**.

Assume $0 < A < C < B$. To determine the cut points $A, B$ and $C$ we solve three indifference equations as follows.




**1 For Player I to be indifferent at $A$:**

(a) If Player I checks at $x = A$, his expected payoff is

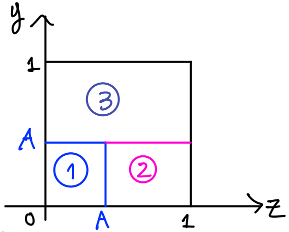
does not matter what II, III does

z,y<x

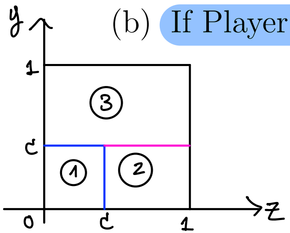
y<x<z

z<x<y

$$\int_0^A \int_0^A 2\,dz\,dy + \int_0^A \int_A^1 -1\,dz\,dy + \int_A^1 \int_0^1 -1\,dz\,dy$$




(b) If Player I bets at $x = A$, his expected payoff is




both fold

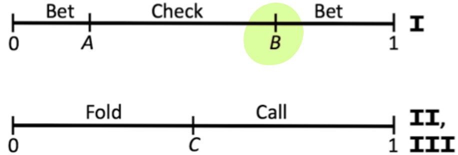
II folds, III calls & wins

II calls & wins

$$\int_0^C \int_0^C 2\,dz\,dy + \int_0^C \int_C^1 -(b+1)\,dz\,dy + \int_C^1 \int_0^1 -(b+1)\,dz\,dy$$

Equating the two expressions above yields the following equation:

$$3A^2 - 1 = 3C^2 + bC^2 - b - 1. \qquad \text{(Eq. A)}$$




**2** **For Player I to be indifferent at $B$:**

(a) If Player I checks at $x = B$, his expected payoff is

does not matter what II, III does

$$\int_0^B \int_0^B 2\,dz\,dy + \int_0^B \int_B^1 -1\,dz\,dy + \int_B^1 \int_0^1 -1\,dz\,dy$$

(b) If Player I bets at $x = B$, his expected payoff is

both fold — fold, call — call, fold — both call

$$\int_0^C \int_0^C 2\,dz\,dy + \int_0^C \int_C^B (b+2)\,dz\,dy + \int_C^B \int_0^C (b+2)\,dz\,dy + \int_C^B \int_C^B 2(b+1)\,dz\,dy$$

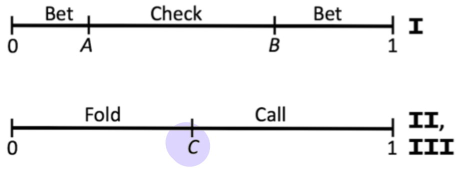
Call — call

$$+ \int_B^1 \int_0^B -(b+1)\,dz\,dy + \int_0^B \int_B^1 -(b+1)\,dz\,dy + \int_B^1 \int_B^1 -(b+1)\,dz\,dy$$

$z<x<y$ — $y<x<z$ — $x<y,z$, both call

Equating the two expressions above yields the following equation:

$$3B^2 - 1 = -2bCB + 3bB^2 + 3B^2 - b - 1. \qquad \text{(Eq. B)}$$




**3 For Player II (or Player III) to be indifferent at $C$:**

Assume that Player I bets.

(a) If Player II folds at $y = C$, her expected payoff is

player I bet

$$\int_0^A \int_0^1 -1\,dz\,dx + \int_B^1 \int_0^1 -1\,dz\,dx$$

(b) If Player II calls at $y = C$, her expected payoff is

Player I bet.

x, z < y

x<y<z

y<x

$$\int_0^A \int_0^C (b+2)\,dz\,dx + \int_0^A \int_C^1 -(b+1)\,dz\,dx + \int_B^1 \int_0^1 -(b+1)\,dz\,dx$$

Equating the two expressions above yields the following equation:

$$-A + B - 1 = 2bCA + 3CA - bA - A - b + bB + B - 1. \qquad \text{(Eq. C)}$$

Solving the above **non-linear** system of three equations in three unknowns gives us the solutions for $A, B$ and $C$ for the Nash equilibrium strategies.

In particular, when $b = 2$, `Optimal(2);` returns:

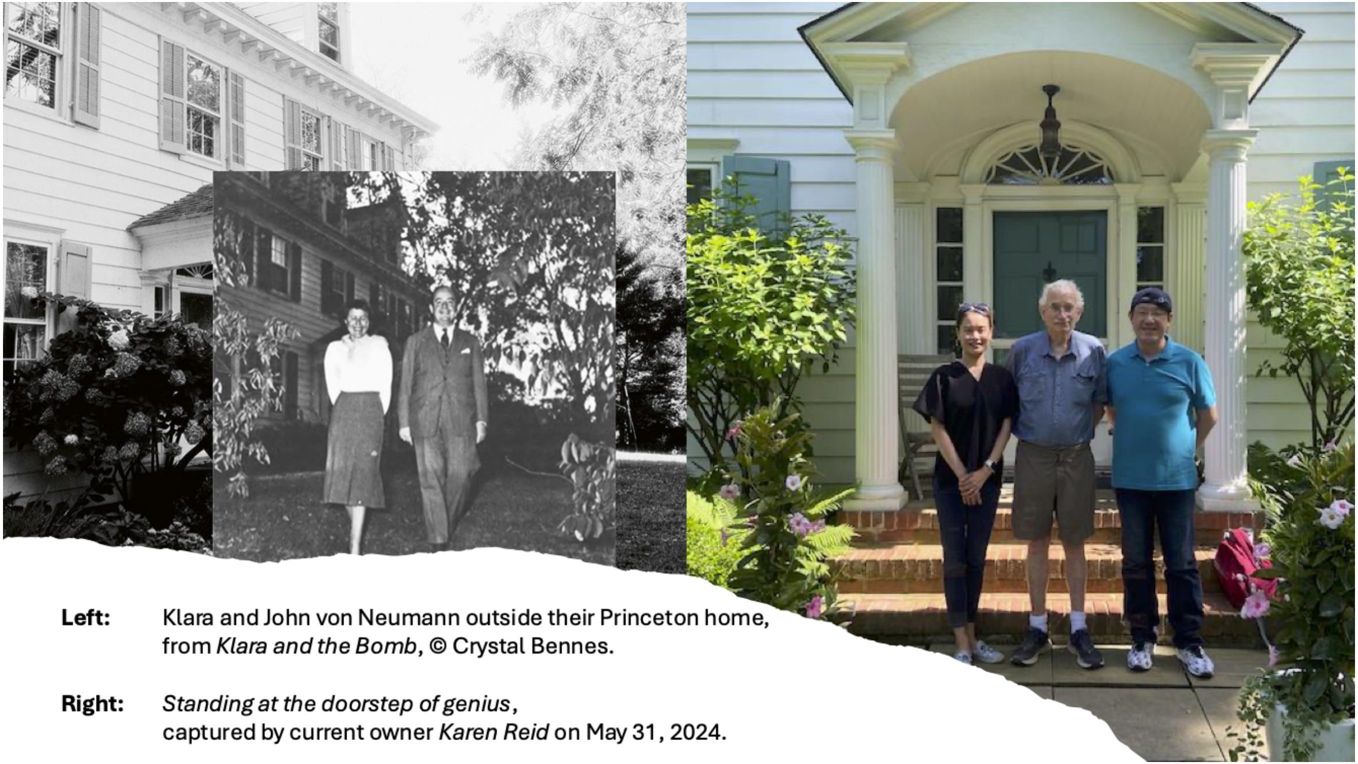
$$
\begin{aligned}
A &= 0.137058194328370 \\
B &= 0.829422249795391 \\
C &= 0.641304115985175.
\end{aligned}
$$

This results in the value of the game (for Player I) being 0.122557074714865.

## Further discussion

- We can also determine the best bet amount $b$, that maximizes Player I's payoff under the Nash equilibrium strategies. Approximately, $b^* \approx 2.07$, resulting in Player I achieving a maximum payoff of 0.122590664136184.
- Therefore, we observe that the highest payoff for Player I in the three-player game exceeds that of the von Neumann's two-player game, which is $1/9 = 0.111111$ achieved at $b^* = 2$.
- Finally, Nash equilibrium for the three-player continuous game resembles those observed in the discrete model when $n$ is large.

# Then & Now: von Neumann's House



**Left:** Klara and John von Neumann outside their Princeton home, from *Klara and the Bomb*, © Crystal Bennes.

**Right:** *Standing at the doorstep of genius*, captured by current owner *Karen Reid* on May 31, 2024.

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# Happy Halloween Eve! *Trick or Treat*