REPORT ON THE PAPER
“THE NUMBER OF INVERSIONS AND THE MAJOR INDEX OF PERMUTATIONS ARE ASYMPTOTICALLY JOINT-INDEPENDENT-NORMAL”
BY A. BAXTER AND D. ZEILBERGER

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I have read the whole paper (Third version, Nov. 4, 2010), and found both the result and the method interesting. I believe that the proof is correct, but there are a few points that I would like to see clarified.

(1) Bottom of P2: the description you give of the existence of a (Gaussian) limit law actually resembles the description of a local limit law (see for instance Flajolet & Sedgewick, Analytic combinatorics, Section IX.9). Even if you carefully explain how $da$, $db$ and $n$ tend to their respective limits, I am not sure that this can be proved using the method of moments. I suggest you stick to the simplest description, that is, the convergence of the distribution function. You may want to give a reference for the method of moments.

(2) The fact that the moments are polynomials in $n$ is essential in your paper. I would like to see the ‘old trick’ mentioned on P3 detailed.

(3) P3, below (OO). Could you explain what symmetry makes (OE) and (EO) true before taking the limits? I think I understand why (OE) is true when $s = 0$ (it follows from the fact that the generating function at the top of P2 is self-reciprocal, right?), but we need a symmetry that involves both statistics to prove it for a generic $s$.

(4) P6, “the leading terms of the FM’s and the Mom’s are the same”. What does it mean? That

$$E_n(X(X - 1)\cdots(X - r + 1)Y(Y - 1)\cdots(Y - s + 1)) \sim E_n(X^r Y^s)$$

for all $r$ and $s$, where $X$ and $Y$ are respectively the centered inversion number and the centered Major index? Is this really true? and why? For instance

$$E_n(X(X - 1)(X - 2)) = -3E_n(X^2) \quad \text{while} \quad E_n(X^3) = 0 \quad \text{by symmetry.}$$

Couldn’t you say instead that that you apply the method of “factorial” moments, proving the convergence of the factorial moments to those of the normal distribution?

(5) Middle of P7: again, it is not obvious to me that $FM(r, s)(n, i)$, for $r$ and $s$ fixed, is a polynomial in $n$ and $i$. Neither from the combinatorics, nor from the recurrences. Are you using a complete version of (Rec$G^1$) and (Gnn$^1$)? This would require more explanations.

(6) I was confused by the first sentence of the paragraph “Nice conjectures but what about proofs?”. I believe there are two reasons for that:

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• it suggests that the paragraph (only) deals with an alternative way of finding the polynomials $F_M(r, s)(n, i)$; as far as I understand, the paragraph is however essential to prove the conjectures.
• since you already used (RecG) and (Gnn) to find many $F_M(r, s)(n, i)$, what will change in this alternative approach is not clear.

(7) Finally, now that I have read Dan Romik’s report, I agree with him that it would be appropriate to cite this nice formula of Roselle (which I did not know, thank you Dan!). It seems to be one way to answer Point (3) above.

Minor remarks, and some typos
• P2 L1, what does “their” refer to?
• P5, I suggest to give the range of validity of (some of) the identities. For instance, one should certainly not apply (RecF) to $i = n$. At the bottom of this page, it could be worth insisting that $F(n + 1, n + 1)(p, q)$ is the generating function of permutations of size $n$ (by (Fnn)), which is why you are especially interested in these polynomials.
• P7 L4, “it is still asymptotically normal”: explain what “it” refers to. On the next line, can you explain where the value of the average comes from? or give a reference?
• P7, It may be worth defining explicitly the numbers $F_M(r, s)(n, i)$ (sorry if I have overlooked their definition!).
• P7, the sentence that contains the expression of $F_M(2r, 2s)$ should be preceded by a period, not a comma. At the same place, make clear that the “degree” is the total degree in $n$ and $i$.
• P8. In (Gnn'), you have kept the term $F_M(r - 1, s - 1)(n - 1, i)$ because it may be of the same order as $F_M(r, s)(n - 1, i)$. But then, shouldn’t you have a term $F_M(r - 1, s - 1)(n - 1, i)$ in (RecG')? I suggest to replace $i$ by $j$ in (Gnn') since it comes from (Gnn). Also, between (RecG') and (Gnn'), replace Gnn by (Gnn).
• La grande finale (du français !). First line: “the case gives”, or “the cases give” (I think...). Recall that you normalize your random variables by $\sigma_n$ (for a while I had forgotten why you should divide by a power of $F_M(0, 2)$). Shouldn’t there be the parameters $(n, i)$ (or $(n + 1, n + 1)$) in the last 4 equations? On the next line, “the mixed moments” is repeated. Finally, replace “the normal distribution $e^{-a^2/2 - b^2/2} / (2\pi)$” by “the normal distribution of density $e^{-a^2/2 - b^2/2} / (2\pi)$”.