Combinatorial Proofs of Capelli's and Turnbull's Identities from Classical Invariant Theory

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0. Introduction

- "Turnbull" is misspelt "Turnbulll" in the first paragraph of the article.

1. The Capelli identity

- In the second line of §1, you write $o_{i,j}$, meaning $p_{i,j}$ obviously.
- When you introduce the $x_{i,j}$ and $p_{i,j}$, it might be good to tell what h is. The important thing is to say that h must commute with all $x_{i,j}$ and all $p_{i,j}$. This is, of course, obvious to anyone who knows where the h comes from, but I am sure Ekhad would have troubles reading the paper without an explicit statement that $hx_{i,j} = x_{i,j}h$ and $hp_{i,j} = p_{i,j}h$...

Combinatorial Proof of Capelli's Identity

- In the sentence "The weight $w(G,K)$ will be defined in the following way: consider the single monomial introduced in (1.3); if i belongs to K, drop $x_{b,i}$ and replace $p_{b_i,i}$ by h; if i belongs to $I \setminus K$, drop $x_{b,i}$ and replace $p_{b_i,i}$ by $x_{b_i,i}\, p_{b_i,i}$.", you should replace $x_{b,i}$ by $x_{b_i,i}$ two times.
- In the formula
$$w(G,K) = \left(\prod_{i \in K} D_i \prod_{i \in I \setminus K} \Delta_i \right) w\,(K)\,,$$
the $w\,(K)$ should be a $w\,(G)$.
- You give two definitions of the term $w\,(G,K)$: **(1)** "The weight $w(G,K)$ will be defined in the following way: consider the single monomial introduced in (1.3); if i belongs to K, drop $x_{b,i}$ and replace $p_{b_i,i}$ by h; if i belongs to $I \setminus K$, drop $x_{b,i}$ and replace $p_{b_i,i}$ by $x_{b_i,i}\, p_{b_i,i}$. Leave the other terms alike." **(2)**
$$w(G,K) = \left(\prod_{i \in K} D_i \prod_{i \in I \setminus K} \Delta_i \right) w\,(K)\,.$$
None of these definitions extends to the symmetric case (i. e. to the proof of Turnbull's Identity). In **(1)**, it becomes unclear whether to drop *all* $x_{b,i}$ (or just one $x_{b,i}$) and to replace *all* $p_{b_i,i}$ (or just one $p_{b_i,i}$) by $x_{b_i,i}\, p_{b_i,i}$. In **(2)**, the differentials $\dfrac{\partial}{\partial p_{b_i,i}} \dfrac{\partial}{\partial x_{b_i,i}}$ in the definition of Δ_i lead to extra coefficients of 2 before the monomials (because the variable $p_{b_i,i}$ may appear twice in the monomial, and

$\dfrac{\partial}{\partial p_{b_i,i}} p_{b_i,i}^2 = 2p_{b_i,i}$). The correct definition that works for both the Capelli and the Turnbull proofs is this here: In order to obtain $w(G,K)$, do the following:

- write out the term $w(G)$ as a product of $x_{i,j}$, $p_{k,\ell}$ and h;
- move all the $x_{i,j}$ to the left, all the $p_{k,\ell}$ to the middle and all the h to the right (so the term looks like $x_{i_1,j_1}x_{i_2,j_2}...x_{i_u,j_u}p_{k_1,\ell_1}p_{k_2,\ell_2}...p_{k_v,\ell_v}$ after the moving) *ignoring* the fact that $x_{i,j}$ and $p_{i,j}$ don't commute (just do as if they commute);
- for each $i \in K$: remove *one* $p_{b_i,i}$ and *one* $x_{b_i,i}$ from the product (which one doesn't matter, since all $x_{i,j}$ commute with each other, and so do all $p_{k,\ell}$), and insert a h at the end of the product.

The resulting term is $w(G,K)$. (Not exactly what *you* call $w(G,K)$, but it has the same value, because $x_{i,j}$ commutes with $p_{k,\ell}$ whenever $(i,j) \neq (k,\ell)$.)

- In many places throughout the text, you are rather inconsistent about whether multiple indexes are to be separated by commata or not. For example, you write: "The simple drop-add rule just defined guarantees that *no* $p_{i,j}$ *remains to the left of* x_{ij} in any of the weight $w(G,K)$."

- You write: "Let $i = i(G,K)$ be the greatest integer $(1 \leq i \leq n-1)$ such that either i a link source belonging to K, or the i-th column has an entry equal to 1 on the last row." Here, "either i" should be "either i is".

- You write: "Hence, as i is in K, but not in K', the operator D_i (resp. Δ_i) is to be applied to $w(G)$ (resp. G') in order to get $w(G,K)$ (resp. $w(G,K')$), so that:" Here, $w(G,K')$ should be $w(G',K')$, and "resp. Δ_i" should be "resp. nothing" (because i is not a link source in G').

- Here is the main issue I am having with this proof: The derivation of (1.6) in the first case. It is morally true, but needs more details in order not to fail in some cases. Your formulae

$$\begin{array}{rlll} |w(G,K)| = & \ldots x_{b_i,a_i}\ h & \ldots p_{b_i,j} & \ldots \\ |w(G',K')| = & \ldots h & \ldots x_{b_i,a_i}\ p_{b_i,j} & \ldots \end{array}$$

are not always correct. The exception is when there is an "*anti-link*" (k,i) in G, by which I mean a pair (k,i) with $i < k$ satisfying $d_i = d_k = 0$ and $(b_k,k) = (b_i,a_i)$. This is almost the same as a link with the only difference that $i < k$ rather than $k < i$. The problem is when $i < k < j$, because in this case this anti-link (k,i) of G gives rise to a link (not anti-link) (k,j) in G', so the number k, which was not a link source in G, becomes one in G', and therefore we need to apply the operator Δ_k to $w(G')$ in order to obtain $w(G',K')$ (while we do *not* have to apply the operator Δ_k to $w(G)$ in order to obtain $w(G,K)$) And as a consequence, in your formulae

$$\begin{array}{rlll} |w(G,K)| = & \ldots x_{b_i,a_i}\ h & \ldots p_{b_i,j} & \ldots \\ |w(G',K')| = & \ldots h & \ldots x_{b_i,a_i}\ p_{b_i,j} & \ldots \end{array}$$

the middle ... terms are not as equal as they look like. And this is not surprising, because these middle terms have a p_{b_i,a_i} in them, so if they were equal, $|w(G,K)|$ and $|w(G',K')|$ would not be equal (because we cannot move the x_{b_i,a_i} to the right past a p_{b_i,a_i})!
Fortunately, this is the *only* problematic case, and in this case the proof needs only a few minor alterations (luckily, there can be only one anti-link with end i). The proof becomes easier when one defines the term $w(G,K)$ the way I did above, because in that case all the $x_{i,j}$ stand before all the $p_{k,\ell}$, so we get

$$\begin{aligned} |w(G,K)| &= \ldots x_{b_i,a_i} \qquad\qquad \ldots p_{b_i,j} \qquad\qquad \ldots\ h\ \ldots \\ |w(G',K')| &= \ldots x_{b_i,a_i} \qquad\qquad \ldots p_{b_i,j} \qquad\qquad \ldots\ h\ \ldots \end{aligned}$$

which are obviously equal (in $|w(G,K)|$, the terms $x_{b_i,i}$ and $p_{b_i,i}$ were removed and replaced by h because of $i \in K$).

- You write: "As before, $w(G)$ and $w(G)$ have opposite signs." The second G here is actually a G'.

- Directly after this sentence, you show that (1.6) holds in the second case, too. I don't think it is necessary - instead it is necessary to show that the mapping $G \mapsto G'$ from the second case into the first one is really the inverse of the mapping $G \mapsto G'$ from the first case into the second one. Once this is shown, (1.6) will clearly hold in the second case because it does in the first case.

- The proof can be generalized almost for free. The generalization is this one:
Let $x_{i,j}$ be mutually commuting indeterminates for all $(i,j) \in \{1,2,...,m\} \times \{1,2,...,n\}$. Let $p_{i,j}$ be mutually commuting indeterminates for all $(i,j) \in \{1,2,...,m\} \times \{1,2,...,n\}$. Assume that $x_{i,j}$ commutes with $p_{k,\ell}$ for all i,j,k,ℓ unless $(i,j) = (k,\ell)$, and assume that $p_{i,j}x_{i,j} - x_{i,j}p_{i,j} = h$ for some h that is independent of i,j and that commutes with all $x_{i,j}$ and with all $p_{i,j}$. For every positive integer n and for $1 \le i,j \le n$ let
$$A_{i,j} = \sum_{k=1}^{m} x_{k,i}p_{k,j} + h(n-i)\,\delta_{i,j}.$$
Let X denote the matrix $(x_{i,j})_{1\le i\le m,\ 1\le j\le n}$, and let P denote the matrix $(p_{i,j})_{1\le i\le m,\ 1\le j\le n}$. Then
$$\begin{aligned} &\sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma)\, A_{\sigma 1,1}A_{\sigma 2,2}...A_{\sigma n,n} \\ &= \sum_{1\le j_1<j_2<...<j_n\le m} \det(\text{the submatrix of } X \text{ consisting of the rows numbered } j_1,\ j_2,\ ...,\ j_n \text{ only}) \\ &\qquad\qquad \cdot \det(\text{the submatrix of } P \text{ consisting of the rows numbered } j_1,\ j_2,\ ...,\ j_n \text{ only}). \end{aligned}$$
(Of course, this means that $\sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma)\, A_{\sigma 1,1}A_{\sigma 2,2}...A_{\sigma n,n} = 0$ for $m<n$ and that $\sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma)\, A_{\sigma 1,1}A_{\sigma 2,2}...A_{\sigma n,n} = \det X \cdot \det P$ for $m=n$.)
The proof is just your proof up to a small correction: In the case $d=0$ (resp. $d_i = 0$), the number b (resp. b_i) should be allowed to range from 1 to m rather than from 1 to n

2. A Combinatorial Proof of Turnbull's Identity.

- In the statement of Turnbull's Identity, you say "their entries satisfying the same commutation rules". Of course, not *literally* the same, because while $x_{i,j}$ commuted with $p_{k,\ell}$ for all i, j, k, ℓ unless $(i,j) = (k,\ell)$ in the Capelli case, in the Turnbull case it has the stronger condition "unless $(i,j) = (k,\ell)$ or $(i,j) = (\ell,k)$". The same remark relates to the antisymmetric analogues.

- In the proof of Turnbull's identity, you misuse the word "verify" in the meaning of "satisfy". (Was it Foata who wrote this part? This "verify instead of satisfy" mistake is a typical error made by Francophones.)

- In the proof of Turnbull's identity, during the construction of K' in Case 2 (on page 8), you write: "Define $K' = K \cup \{(i,j)\}$ in the first subcase and $K' = K \cup \{(i,j),(k,i)\} \setminus \{(k,j)\}$." Though it is clear what you want to say here, it wouldn't hurt to add "in the second one" at the end of this sentence.

- Just as the Capelli identity, the Turnbull identity can be generalized:
  Let m and n be integers such that $m \geq n \geq 0$. Let $x_{i,j}$ be mutually commuting indeterminates for all $(i,j) \in \{1,2,...,m\} \times \{1,2,...,n\}$ bound only to the relations
  $$(x_{i,j} = x_{j,i} \text{ for all } (i,j) \in \{1,2,...,n\} \times \{1,2,...,n\}).$$
  Let $p_{i,j}$ be mutually commuting indeterminates for all $(i,j) \in \{1,2,...,m\} \times \{1,2,...,n\}$ bound only to the relations
  $$(p_{i,j} = p_{j,i} \text{ for all } (i,j) \in \{1,2,...,n\} \times \{1,2,...,n\}).$$
  Assume that $x_{i,j}$ commutes with $p_{k,\ell}$ for all i,j,k,ℓ unless $\{i,j\} = \{k,\ell\}$, and assume that $p_{i,j}x_{i,j} - x_{i,j}p_{i,j} = h$ for some h that is independent of i,j and that commutes with all $x_{i,j}$ and with all $p_{i,j}$. For every positive integer n and for $1 \leq i,j \leq n$ let
  $$A_{i,j} = \sum_{k=1}^{m} x_{k,i}\widetilde{p}_{k,j} + h\,(n-i)\,\delta_{i,j},$$
  where $\widetilde{p}_{k,j}$ is defined as $p_{k,j}\,(1+\delta_{k,j})$.
  Let X denote the matrix $(x_{i,j})_{1\leq i\leq m,\ 1\leq j\leq n}$, and let $\widetilde{P}$ denote the matrix $(p_{i,j}\,(1+\delta_{i,j}))_{1\leq i\leq m,\ 1\leq j\leq n}$. Then
  $$\begin{aligned}&\sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma)\, A_{\sigma 1,1}A_{\sigma 2,2}...A_{\sigma n,n}\\ &= \sum_{1\leq j_1<j_2<...<j_n\leq m} \det(\text{the submatrix of } X \text{ consisting of the rows numbered } j_1,\ j_2,\ ...,\ j_n \text{ only})\\ &\qquad \cdot \det\left(\text{the submatrix of } \widetilde{P} \text{ consisting of the rows numbered } j_1,\ j_2,\ ...,\ j_n \text{ only}\right).\end{aligned}$$
  (Of course, this means that $\sum\limits_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma)\, A_{\sigma 1,1}A_{\sigma 2,2}...A_{\sigma n,n} = \det X \cdot \det \widetilde{P}$ for $m = n$.)
  The proof is just your proof up to a small correction: In the case $d = 0$ (resp. $d_i = 0$), the number b (resp. b_i) should be allowed to range from 1 to m rather

than from 1 to n.
Note that *this generalization easily yields the formula (11.2.7) in your reference [H-U]* (while the Turnbull identity itself doesn't).

3. What about the Anti-symmetric Analog?

- In the statement of the Howe-Umeda-Kostant-Sahi Identity, I bet you want $x_{i,i} = 0$ and not only $x_{i,j} = -x_{j,i}$. (Or are you working over a field of characteristic 0 all the time? It is not quite clear.)
- A formula is labelled (1") on page 9. This seems out of place; probably (3.1) would be more appropriate.
- In the statement of Turnbull's Anti-Symmetric Analog, you write "be an anti-symmetric matrices". The "an" is misplaced here.

References

- The right page numbers for [T] are p. 76-86.