On a Conjecture of Melkamu Zeleke

Shalosh B. EKHAD

In his fascinating talk[Z], Melkamu Zeleke made the following conjecture.

Conjecture: Let $C(z)$ be the generating function of the Catalan numbers, i.e. the formal power series satisfying

$$C(z) = 1 + zC(z)^2 \quad ,$$

then for all positive integers m, the formal power series

$$F_m(z) := \frac{zC^2 - (zC^2)^m}{1 - (zC^2)^{m+1}} \quad ,$$

are all **rational functions** of z.

In this short note we will give an explicit description of $F_m(z)$ as a rational function, that, once conjectured, can be **rigorously** proved by **only** checking finitely many special cases. It turns out that checking it for $2 \leq m \leq 10$ suffices! (why?). Just to play it safe I checked the proposition below for $2 \leq m \leq 100$.

Proposition Let $N_m(z)$ be the coefficient of X^m in the Maclaurin expansion of

$$\frac{-z - zX + z^2X^2}{1 + (1 - 2\,z)\,X^2 + z^2X^4} \quad ,$$

and Let $D_m(z)$ be the coefficient of X^m in the Maclaurin expansion of

$$\frac{-1 + z + (2\,z - 1)\,X - z^2X^2 - z^2X^3}{1 + (1 - 2\,z)\,X^2 + z^2X^4} \quad ,$$

then

$$F_m(z) = \frac{N_{m-2}(z)}{D_{m-2}(z)} \quad . \quad \square$$

Reference

[Z] Melkamu Zeleke, *On Subsets of Ordered Trees Enumerated by a Subsequence of Fibonacci Numbers* , talk delivered at the Rutgers Experimental Mathematics Seminar, April 18, 2013.

(Based on joint work with Mahendra Jani)

Shalosh B. Ekhad, Mathematics Department, Rutgers University (New Brunswick), Piscataway, NJ 08854, USA. c/o `zeilberg at math dot rutgers dot edu`

April 19, 2013