A note on the paper : Closed Form (pun intended),1993 by Doron Zeilberger

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1.**Definition**: A WZ-form of degree k , k an integer, is a closed difference form that can be written as

$$w = f(m_1, ..., m_n)(\sum_I P_I(m_1, .., m_n)\delta_I)$$

where $P_I(m_1, ..m_n)$ is a polynomial in the variables $m_1, ...m_n$, and $f(m_1, ..., m_n)$ is a closed form.(i.e. $\delta f(m_1, .., m_n) = 0$).

In this paper , Closed Form(pun intended, 1993) several examples of WZ forms of degree one are given. Eq.(7.6) in the paper as stated is not closed just for what is assumed in eq. (7.5). But it might be possible under special cases it turns out to be closed. What is more general under the assumption in equation (7.5) is that one can get $\binom{r}{2}$ possible WZ-forms of degree one from the r parameters :

$$w_i = F(n_1, n_2, ..., n_r, k)\delta k + G_i(n_1, n_2, ..., n_r, k)\delta n_i, i = 1..r$$

To find the corresponding G_i we apply Zeilberger's algorithm to $F(n_1, ..., m_r, k)$ with respect to each variable m_i and find the WZ-mate G_i.
Using the above procedure for Dixon's identity,

$$\sum_k (-1)^k \frac{(a+b)!(a+c)!(b+c)!a!b!c!}{(2(a+k)!(a-k)!(b+k)!(b-k)!(c+k)!(c-k)!(a+b+c)!} = 1$$

we get the following three of the possible six WZ-forms of degree one.

$$W_{Dixon} = (-1)^k \frac{(a+b)!(a+c)!(b+c)!a!b!c!}{(2(a+k)!(a-k+1)!(b+k)!(b-k+1)!(c+k)!(c-k+1)!(a+b+c+1)!}$$

$(2(a-k+1)(b-k+1)(c-k+1)(a+b+c+1)\delta k + (b-k+1)(b+k)(c-k+1)(c+k)\delta a)$.

$$W_{Dixon} = (-1)^k \frac{(a+b)!(a+c)!(b+c)!a!b!c!}{(2(a+k)!(a-k+1)!(b+k)!(b-k+1)!(c+k)!(c-k+1)!(a+b+c+1)!}$$

$(2(a-k+1)(b-k+1)(c-k+1)(a+b+c+1)\delta k + (a-k+1)(a+k)(c-k+1)(c+k)\delta b)$.

$$W_{Dixon} = (-1)^k \frac{(a+b)!(a+c)!(b+c)!a!b!c!}{(2(a+k)!(a-k+1)!(b+k)!(b-k+1)!(c+k)!(c-k+1)!(a+b+c+1)!}$$

$(2(a-k+1)(b-k+1)(c-k+1)(a+b+c+1)\delta k + (a-k+1)(a+k)(b-k+1)(b+k)\delta c)$.

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on the other hand there are no G_a, G_b and G_c such that

$$(\mathbf{A}-\mathbf{1})F(k,a,b,c) = (\mathbf{B}-\mathbf{1})G_a(k,a,b,c)$$

$$(\mathbf{A}-\mathbf{1})F(k,a,b,c) = (\mathbf{C}-\mathbf{1})G_c(k,a,b,c)$$

$$(\mathbf{B}-\mathbf{1})F(k,a,b,c) = (\mathbf{C}-\mathbf{1})G_c(k,a,b,c)$$

Where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are forward shift operator in a,b and c respectively by 1 and $\mathbf{1}$ is the identity operator. Even in his case the discussion in eq. 7.3 shows an operator,say $R(A,a)$, can exist such that(similarly for the remaining cases)

$$(A-\mathbf{1})(R(A,a)F(k,a,b,c)) = (B-\mathbf{1})G_b(k,a,b,c)$$

Which then gives a wz form of degree 1

$$w = R(A,a)F(k,a,b,c)\delta a + G_b(k,a,b,c)\delta b$$

In the same way,from Saalschut's identity

$$\sum_k \frac{(m-r+s)!(n+r-s)!(r+k)!m!n!(r-m)!(s-n)!}{k!(m-r+s-k)!(n-k)!(r-s+k)!(m+n)!(r+k-m-n)!r!s!} = 1$$

we get four wz-forms of degree one.

$$W_{Saalscutz's} = \frac{(m-r+s)!(n+r-s)!(r+k)!(m!n!(r-m)!(s-n)!}{k!(m-r+s-k)!(n-k)!(r-s+k)!(m+n)!(r+k-m-n)!r!s!(n-k+1)}.$$

$\frac{1}{(m+n+1)(n-s)}((n-k+1)(m+n+1)(n-s)\delta k + -k(r-s+k)(-r-k+m+n)\delta n)$

$$W_{Saalscutz's} = \frac{(m-r+s)!(n+r-s)!(r+k)!(m!n!(r-m)!(s-n)!}{k!(m-r+s-k)!(n-k)!(r-s+k)!(m+n)!(r+k-m-n)!r!s!(r+1)(-m+r-s)}.$$

$((r+1)(-m+r-s)\delta k + (-2r-m+1-s-k)k\delta r)$

$$W_{Saalscutz's} = \frac{(m-r+s)!(n+r-s)!(r+k)!(m!n!(r-m)!(s-n)!}{k!(m-r+s-k)!(n-k)!(r-s+k)!(m+n)!(r+k-m-n)!r!s!(m-r+s-k+1)}.$$

$\frac{1}{(s+1)(-n-r+s)}((m-r+s-k+1)(s+1)(-n-r+s)\delta k - k(-r+s-k)(r+k-m-n)\delta s)$

$$W_{Saalscutz's} = \frac{(m-r+s)!(n+r-s)!(r+k)!(m!n!(r-m)!(s-n)!}{k!(m-r+s-k)!(n-k)!(r-s+k)!(m+n)!(r+k-m-n)!r!s!(m-r)}$$

$\frac{1}{(m-r+s-k+1)(m+n+1)}((m-r+s-k+1)(m+n+1)(m-r)\delta k - k(r-s+k)(-r-$
$k+m+n)\delta m)$

2. The generalization given after the proof of Theorem 9 is also seen in the same line as above and the discussion following reframed. i.e. one can start with a known multi-variables

$$\sum C(n, k_1, ..., k_r) = cont.$$

and get $\binom{r}{2}$ additional potential function for a WZ-1 forms.

$$w = F(n, k_1, ..., k_r)\delta n + \sum_{i=1}^{r} G_i(n, k_1, ..., k_r)\delta k_i$$

to be closed we need stronger general conditions than equation (13.21).

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