ON FRAENKEL'S N-HEAP WYTHOFF'S CONJECTURES

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Abstract. The N-heap Wythoff's game is a two-player impartial game with N piles of tokens of sizes $A^1, \ldots, A^N$, $A^1 \leq \cdots \leq A^N$. Players take turns removing any number of tokens from a single pile, or removing $(a_1, \ldots, a_N)$ from all piles — a_i tokens from the i-th pile, providing that $0 \leq a_i \leq A^i$, $\sum_{i=1}^{N} a_i > 0$ and $a_1 \oplus \cdots \oplus a_N = 0$, where $\oplus$ is the nim addition. The first player that cannot make a move loses. Denote all the P-positions (i.e., losing positions) by $(A^1, \ldots, A^{N-2}, A_n^{N-1}, A_n^N)$, $A^{N-2} \leq A_n^{N-1} \leq A_n^N$ and $A_n^{N-1} < A_{n+1}^{N-1}$ for all $n \geq 1$. Two conjectures were proposed on the game by Fraenkel [7]. When $A^1, \ldots, A^{N-2}$ are fixed, i) there exists an integer N_1 such that when $n > N_1$, $A_n^N = A_n^{N-1} + n$. ii) there exist integers N_2 and α_2 such that when $n > N_2$, $A_n^{N-1} = \lfloor n\phi \rfloor + \epsilon_n + \alpha_2$ and $A_n^N = A_n^{N-1} + n$, where $-1 \leq \epsilon_n \leq 1$ and $\phi = (1+\sqrt{5})/2$, the golden section.

In this paper, we provide a sufficient condition for the conjectures to hold, and subsequently prove them for the three-heap Wythoff's game with the first piles having up to 10 tokens.

1. Introduction

An impartial game in the normal play is a game in which two players take turns making moves, and the first player who cannot make a move loses. All the information about the game must be available to both players (e.g., unlike most popular card games); all moves must be accessible to both of them; and there is no chance moves (e.g., no dice); and the outcome must be win or lose. Under the assumption of "perfect play" (the two players are infinitely smart), we say that a position is a P-position if the player who makes the *previous* move will win, otherwise it is an N-position, i.e., the *next* player will win.

Wythoff's Game [13] is an impartial game consisting of two piles of tokens. Players can remove any number of tokens from a single pile, or the same number of tokens from both piles. The P-positions are well-known and well explained by Fraenkel [6]: they are a sequences of pairs of integers $\{(A_n, B_n)\}_{n \geq 0}$, such that $A_n = mex\{A_m,\ B_m \mid 0 \leq m < n\}$ and $B_n = A_n + n$ with $A_0 = B_0 = 0$, where mex is the *M*inimal *EX*clusive value, i.e., the least nonnegative integer that is not in the set. They can also be written as $A_n = \lfloor n\phi \rfloor$ and $B_n = \lfloor n\phi^2 \rfloor$, where $\phi = (1+\sqrt{5})/2$ (the golden section). Various generalizations and results on this game were done by Blass and Fraenkel [1], Blass, Fraenkel, Guelman [2], WW [3], Coxeter [4], Dress [5],

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Fraenkel and Borosh [8], Fraenkel and Ozery [9], Fraenkel and Zusman [10], Landman [12], Yaglom and Yaglom [14].

Another generalization of Wythoff's game, involving more than two piles, was proposed by Fraenkel [7], which is listed in the survey article by Guy and Nowakowski [11] as one of the "unsolved problems in combinatorial games". We are given N piles of tokens, whose sizes are $A^1, \ldots, A^N$, $A^1 \leq \cdots \leq A^N$. A player can remove any number of tokens from a single pile, or, for any non-zero vector of non-negative integers $(a_1, \ldots, a_N)$ whose *nim-sum* is 0, remove a_i tokens from the i-th pile (for $1 \leq i \leq N$). [Recall that the *nim-sum* (denoted by $\oplus$) is binary addition without carry. For example $3 \oplus 5$ equals $011 \oplus 101 = 110 = 6$.]

Denote all the P-positions by

$$(A^1, \ldots, A^{N-2}, A_n^{N-1}, A_n^N), \quad , A^{N-2} \leq A_n^{N-1} \leq A_n^N$$

and

$$A_n^{N-1} < A_{n+1}^{N-1} \quad for \quad all \quad n \geq 0 \quad .$$

Fraenkel's conjectures are as follows. Fix $A^1, \ldots, A^{N-2}$, then

Conjecture 1: There exists an integer N_1 (depending only on $A^1, \ldots, A^{N-2}$), such that when $n > N_1$, $A_n^N = A_n^{N-1} + n$.

Conjecture 2: There exist integers N_2 and α_2 such that when $n > N_2$, $A_n^{N-1} = \lfloor n\phi \rfloor + \epsilon_n + \alpha_2$ and $A_n^N = A_n^{N-1} + n$, where $\phi = (1 + \sqrt{5})/2$ and $-1 \leq \epsilon_n \leq 1$.

Furthermore, $A_n^{N-1} = mex(\{A_i^{N-1}, A_i^N : \ 0 \leq i < n\} \ \cup \ T)$, where T is a (small) set of integers.

In this paper, we prove the conjectures for the three-heap Wythoff's game when $A^1 \leq 10$.

2. Preliminary Results

Throughout this paper, we use the notation $\phi = (1 + \sqrt{5})/2$, the golden section.

Definition 1. We call a sequence of pairs of integers $\{(A_n, B_n)\}_{n \geq n_0}$ a *Wythoff's sequence* if there exist a finite set of integers T such that $A_n = mex(\{A_i, B_i : n_0 \leq i < n\} \cup T)$, $B_n = A_n + n$ and $\{B_n\} \cap T = \emptyset$.

Definition 2. A *special Wythoff's sequence* is a Wythoff's sequence such that there exist integers N and α such that when $n > N$, $A_n = \lfloor n\phi \rfloor + \alpha + \epsilon_n$, where $\epsilon_n \in \{0, \pm 1\}$.

Lemma 2.1. *Given a Wythoff's sequence* $\{(A_n, B_n)\}_{n \geq n_0}$ *and* $N \geq n_0$ *such that* $A_n > \max(T)$ *for all* $n \geq N$, *then*

(1) $1 \leq A_{n+1} - A_n \leq 2$,
(2) $2 \leq B_{n+1} - B_n \leq 3$, *and*

(3) *if* $A_n > B_N$, $A_{n+2} - A_n \geq 3$.

Proof: By definition, $A_n - A_{n-1} \geq 1$, $B_n - B_{n-1} = A_n + n - A_{n-1} - (n-1) \geq 2$. Also, since $\{A_i\}_{i \geq n_0} \cup \{B_i\}_{i \geq n_0} = \mathbb{Z} - T$ and $A_n > \max(T)$, the only numbers between A_n and A_{n+1} are in $\{B_i\}_{i \geq n_0}$, which are not pair-wise sequential, therefore $A_{n+1} - A_n \leq 2$ and $B_n - B_{n-1} = A_n - A_{n-1} + 1 \leq 3$. Furthermore, if $A_{n+2} - A_n = 2$, let $B_m = \min\{B_i : B_i > A_{n+2}\}$, then $m > N$, $B_{m-1} < A_n$, so $B_m - B_{m-1} > 3$, which is contradictory to what we just proved.

Lemma 2.2. *$\{(\lfloor n\phi \rfloor, \lfloor n\phi \rfloor + n)\}_{n \geq 1}$, the Wythoff's pairs, is a special Wythoff's sequence. In addition to all the properties in Lemma 2.1, it also satisfies the following:*

(1) $A = \{\lfloor n\phi \rfloor\}_{n \geq 1}$ *and* $B = \{n + \lfloor n\phi \rfloor\}_{n \geq 1}$ *are complementary, i.e.,* $A \cup B = \mathbb{Z}_{>0}$ *and* $A \cap B = \emptyset$,
(2) $1 \leq \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor \leq 2$,
(3) $|\lfloor n_1\phi \rfloor - \lfloor n_2\phi \rfloor - (n_1 - n_2)\phi| < 1$,
(4) *if* $\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = 1$, *then* $\lfloor (n+1)\phi \rfloor - \lfloor n\phi \rfloor = \lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor = 2$,
(5) *if* $\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = \lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor = 2$, *then* $\lfloor (n+1)\phi \rfloor - \lfloor n\phi \rfloor = \lfloor (n-2)\phi \rfloor - \lfloor (n-3)\phi \rfloor = 1$, *and*

Proof: It is well explained by Fraenkel [6] that the sequence satisfies all the requirements for special Wythoff's sequence.

(1) See in [6].
(2) is a direct corollary of the previous lemma. The first two items are highlighted again because they are of special interest to us in the coming sections.
(3) Since ϕ is irrational, $n_1\phi - 1 - n_2\phi < \lfloor n_1\phi \rfloor - \lfloor n_2\phi \rfloor < n_1\phi - n_2\phi + 1$.
(4) Since $\phi \approx 1.618$, for any n, $\lfloor n\phi \rfloor - \lfloor (n-2)\phi \rfloor > n\phi - 1 - (n-2)\phi > 2.2$. Since the left-hand side of the inequality is an integer, it is at least 3.
(5) Similarly for any n, $\lfloor n\phi \rfloor - \lfloor (n-3)\phi \rfloor < n\phi - (n-3)\phi + 1 < 5.9$. So the left-hand side can be at most 5.

Lemma 2.3. *Given a Wythoff's sequence* $\{(A_n, B_n)\}_{n \geq n_0}$ *and* $N \geq n_0$ *such that* $A_n > \max(T)$ *for all* $n \geq N$, *if we write* $A_n = \lfloor n\phi \rfloor + \alpha_n$ *and* $B_n = A_n + n + \alpha_n$, *then* $-1 \leq \alpha_{n+1} - \alpha_n \leq 1$ *for all* $n \geq N$.

Proof: $\alpha_{n+1} - \alpha_n = (A_{n+1} - A_n) - (\lfloor (n+1)\phi \rfloor - \lfloor n\phi \rfloor)$, so the inequality holds because of Lemma 2.1 and Lemma 2.2.

Lemma 2.4. *Given a Wythoff's sequence* $\{(A_n, B_n)\}_{n \geq n_0}$, *we assume that there exist integers* N, α, *and* $m_2 > m_1 > N + 2$, *such that*

- $A_n > \max(T)$, *when* $n > N$,
- $A_{m_2} > B_{m_1}$,
- $A_n = \lfloor n\phi \rfloor + \alpha + \epsilon_n$, *where* $m_1 - 2 \leq n \leq m_2$ *and* $-1 \leq \epsilon_n \leq 1$, *and*

- $\epsilon_{i-2}\epsilon_{i-1}\epsilon_i = 0$, *when* $m_1 - 2 \leq i \leq m_2$,

then $\epsilon_{i-1}\epsilon_i = 0$ *for any* $i > m_2$.

Proof: The last assumption is equivalent to the statement that there do not exist three consecutive non-zero ϵ's by Lemma 2.3 when $m_1 - 2 \leq i \leq m_2$, and we are to prove 1) $-1 \leq \epsilon_i \leq 1$ when $i > m_2$; and 2) there are no two consecutive non-zero ϵ's when $i > m_2$. Once proved, this lemma provides a way to evaluate the behavior of ϵ_n, and *when* the behavior starts.

We are going to prove the lemma in two steps:

First, if $\exists n > m_2$ so that $\epsilon_n \notin \{0, \pm 1\}$, let n be the smallest of such numbers, then $\epsilon_n = \pm 2$ by Lemma 2.3. There are four cases:

a) $A_n = A_{n-1} + 2$ and $\epsilon_n = -2$: $\epsilon_{n-1} = -1$ by Lemma 2.3, and $A_n = \lfloor n\phi \rfloor + \alpha - 2 \leq \lfloor (n-1)\phi \rfloor + \alpha$ by Lemma 2.2. However, $A_n = A_{n-1} + 2 = \lfloor (n-1)\phi \rfloor + \alpha + 1$, which is impossible.

b) $A_n = A_{n-1} + 2$ and $\epsilon_n = 2$: $\epsilon_{n-1} = 1$ by Lemma 2.3, so $\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = A_n - A_{n-1} - (\epsilon_n - \epsilon_{n-1}) = 1$. By Lemma 2.2, $\lfloor (n+1)\phi \rfloor - \lfloor n\phi \rfloor = \lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor = 2$. Since $A_n = A_{n-1} + 2$, there exists $m < n$ so that $B_m = A_n - 1 = A_{n-1} + 1$, i.e., $\lfloor m\phi \rfloor + m + \epsilon_m = \lfloor n\phi \rfloor + 1$. Since $\{\lfloor n\phi \rfloor + n\}$ and $\{\lfloor n\phi \rfloor\}$ are complementary and $\lfloor n\phi \rfloor = \lfloor (n+1)\phi \rfloor - 2$, we must have $\epsilon_m = 0$ and $\lfloor m\phi \rfloor + m = \lfloor n\phi \rfloor + 1$. Similarly since $\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = 1$, $\lfloor (m-1)\phi \rfloor + m - 1 = \lfloor (n-1)\phi \rfloor - 1 = \lfloor m\phi \rfloor + m - 3$, thus $\lfloor m\phi \rfloor = \lfloor (m-1)\phi \rfloor + 2$. Also, since $2 \geq A_{n-1} - A_{n-2} = \lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor + \epsilon_{n-1} - \epsilon_{n-2} = 3 - \epsilon_{n-2}$, $\epsilon_{n-2} = 1$ and $A_{n-1} - A_{n-2} = 2$. By our assumption of $\epsilon_{n-1}\epsilon_{n-2}\epsilon_{n-3} = 0$, we know $\epsilon_{n-3} = 0$, and by the definitions of A_n and B_n, $B_{m-1} = A_{n-1} - 1$, which is to say $\lfloor (m-1)\phi \rfloor + m - 1 + \epsilon_{m-1} = \lfloor (n-1)\phi \rfloor$. Again, since $\{\lfloor n\phi \rfloor + n\}$ and $\{\lfloor n\phi \rfloor\}$ are complementary and $\lfloor n\phi \rfloor = \lfloor (n-1)\phi \rfloor + 1$, ϵ_{m-1} has to be 1.

Now consider $A_{n-2} - A_{n-3}$, which is 1 or 2 by Lemma 2.1. If $A_{n-2} - A_{n-3} = 1$, we have $1 = A_{n-2} - A_{n-3} = \lfloor (n-2)\phi \rfloor - \lfloor (n-3)\phi \rfloor + 1 - \epsilon_{n-3}$, which means $\lfloor (n-2)\phi \rfloor - \lfloor (n-3)\phi \rfloor = 1$ and $\epsilon_{n-3} = 1$. But then $\epsilon_{n-1}\epsilon_{n-2}\epsilon_{n-3} = 1$, which is contradictory to our assumption. If $A_{n-2} - A_{n-3} = 2$, $B_{m-2} = A_{n-2} - 1$, which means $\lfloor (m-2)\phi \rfloor + m - 2 + \epsilon_{m-2} = \lfloor (n-2)\phi \rfloor$. Again, since $\{\lfloor n\phi \rfloor + n\}$ and $\{\lfloor n\phi \rfloor\}$ are complementary and $\lfloor (n-2)\phi \rfloor - \lfloor (n-3)\phi \rfloor = A_{n-2} - A_{n-3} - (\epsilon_{n-2} - \epsilon_{n-3}) = 1$, ϵ_{m-2} has to be -1. But $\epsilon_{m-1} = 1$, contradictory to Lemma 2.3.

c) $A_n = A_{n-1} + 1$ and $\epsilon_n = 2$: By Lemma 2.3, $\epsilon_{n-1} = 1$, thus $\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = A_n - A_{n-1} - \epsilon_n + \epsilon_{n-1} = 0$, which is impossible.

d) $A_n = A_{n-1} + 1$ and $\epsilon_n = -2$: By Lemma 2.1, $A_{n-1} = A_{n-2} + 2$, and by Lemma 2.3, $\epsilon_{n-1} = -1$. Therefore $-1 \leq \epsilon_{n-2} = \lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor - 3$, so $\epsilon_{n-2} = -1$, $\lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor = 2$, and $\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = A_n - A_{n-1} - (\epsilon_n - \epsilon_{n-1}) = 2$ thus $\lfloor (n-2)\phi \rfloor - \lfloor (n-3)\phi \rfloor = 1$

by Lemma 2.2. However $1 \le A_{n-2} - A_{n-3} = \lfloor (n-2)\phi \rfloor - \lfloor (n-3)\phi \rfloor - 1 - \epsilon_{n-3}$, which means $\epsilon_{n-3} = -1$ and $\epsilon_{n-1}\epsilon_{n-2}\epsilon_{n-3} = -1$, contradictory to our assumption.

Secondly, if there exists $n > m_2$ so that $\epsilon_n \epsilon_{n-1} \neq 0$, i.e., $\epsilon_n = \epsilon_{n-1} = \pm 1$, let n be the smallest of such numbers, then by Lemma 2.1 and by what we have just proved, $\epsilon_{n-2} = 0$. There are two cases:

e) $\epsilon_n = \epsilon_{n-1} = 1$: $2 \ge A_{n-1} - A_{n-2} = \lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor + 1$, so $\lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor = 1$ and $A_{n-1} - A_{n-2} = 2$. Also $A_n - A_{n-1} = \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = 2$ by Lemma 2.2. So there exists $m < n$ such that $B_m = A_n - 1$ and $B_{m-1} = A_{n-1} - 1$, which means $\lfloor m\phi \rfloor + m + \epsilon_m = \lfloor n\phi \rfloor$ and $\epsilon_m = \pm 1$; $\lfloor (m-1)\phi \rfloor + m - 1 + \epsilon_{m-1} = \lfloor (n-1)\phi \rfloor$ and $\epsilon_{m-1} = \pm 1$. Since $\{\lfloor n\phi \rfloor + n\}$ and $\{\lfloor n\phi \rfloor\}$ are complementary and $\lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor = 1$, $\epsilon_{m-1} = -1$, therefore $\epsilon_m = \epsilon_{m-1} = -1$ by Lemma 2.3, and $\lfloor m\phi \rfloor + m = \lfloor n\phi \rfloor + 1$. So $\lfloor (n+1)\phi \rfloor = \lfloor n\phi \rfloor + 2$, and $\lfloor (n+2)\phi \rfloor = \lfloor (n+1)\phi \rfloor + 1$ by Lemma 2.2. Thus $\lfloor (m+1)\phi \rfloor + m + 1 = \lfloor (n+2)\phi \rfloor + 1 = \lfloor m\phi \rfloor + m + 3$, and $3 \ge B_{m+1} - B_m = \lfloor (m+1)\phi \rfloor + m + 1 + \epsilon_{m+1} - (\lfloor m\phi \rfloor + m - 1) = 4 + \epsilon_{m+1}$. Therefore $\epsilon_{m+1} = -1$ and $\epsilon_{m+1}\epsilon_m\epsilon_{m-1} = -1$, contradictory to our assumption.

f) $\epsilon_n = \epsilon_{n-1} = -1$: $1 \le A_{n-1} - A_{n-2} = \lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor - 1$, so $\lfloor (n-1)\phi \rfloor - \lfloor (n-2)\phi \rfloor = 2$ and $A_{n-1} - A_{n-2} = 1$. Thus $A_n - A_{n-1} = A_{n-2} - A_{n-3} = 2$ by Lemma 2.1, and $\lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor = A_n - A_{n-1} = 2$. Hence $\lfloor (n+1)\phi \rfloor - \lfloor n\phi \rfloor = \lfloor (n-2)\phi \rfloor - \lfloor (n-3)\phi \rfloor = 1$ by Lemma 2.2. Therefore there exists $m < n$ such that $B_m = A_n - 1$, $B_{m-1} = A_{n-2} - 1$, that is $\lfloor n\phi \rfloor - 2 = \lfloor m\phi \rfloor + m + \epsilon_m$, and $\lfloor (m-1)\phi \rfloor + m - 1 + \epsilon_{m-1} = \lfloor (n-2)\phi \rfloor - 1$. Since $\{\lfloor n\phi \rfloor + n\}$ and $\{\lfloor n\phi \rfloor\}$ are complementary and $\lfloor n\phi \rfloor = \lfloor (n-1)\phi \rfloor + 2$, $\epsilon_m = \pm 1$. Similarly, since $\lfloor (n-2)\phi \rfloor - \lfloor (n-3)\phi \rfloor = 1$, $\epsilon_{m-1} = 1$, which means $\epsilon_m = \epsilon_{m-1} = 1$ by Lemma 2.3. So $\lfloor m\phi \rfloor + m = B_m - 1 = A_n - 2 = \lfloor n\phi \rfloor - 3$, and $\lfloor (m+1)\phi \rfloor + m + 1 = \lfloor n\phi \rfloor - 1$, thus $\lfloor (m+1)\phi \rfloor = \lfloor m\phi \rfloor + 1$. Since $2 \le B_{m+1} - B_m = 1 + \epsilon_{m+1}$, $\epsilon_{m+1} = 1$ and $\epsilon_{m+1}\epsilon_m\epsilon_{m-1} = 1$, contradictory to our assumption.

3. Main Results

In this section, we adopt the following notation:

- $[a, b, c]$, $a \ge 0$, $b \ge 0$, $c \ge 0$, is a three-heap Wythoff's position having a, $a+b$ and $a+b+c$ tokens in the piles;
- $[m, A_n^m, B_n^m]$, where $n \ge 1$ and $A_n^m < A_{n+1}^m$, are all the P-positions whose first piles have m tokens;
- P^m is the set of P-positions whose first piles have m tokens;
- $T^m = \mathbb{Z}_{\ge 0} - \{A_i^m, A_i^m + B_i^m : i \ge 1\} - \{i : 0 \le i < m\}$;
- $S^m = \mathbb{Z}_{\ge 0} - \{B_i^m : i \ge 1\}$;
- $\alpha_n^m = m + A_n^m - \lfloor B_n^m \phi \rfloor$;
- N_1^m is the integer such that when $n > N_1^m$, $A_n^m = mex\{A_i^m, B_i^m : 1 \le i < n\}$ and $B_{n+1}^m = B_n^m + 1$;

- N_2^m, α^m, and ϵ_n^m are the integers such that when $n > N_2^m$; $A_n^m = mex\{A_i^m, B_i^m : 1 \leq i < n\}$, $B_{n+1}^m = B_n^m + 1$; and $\epsilon_n^m = m + A_n^m - \lfloor B_n^m \phi \rfloor - \alpha^m \in \{0, \pm 1\}$;
- N_3^m is the integer such that if N_2^m exists and when $n > N_3^m$, $\epsilon_n^m \epsilon_{n+1}^m = 0$;
- $p(m) = 2^{\lfloor \log_2(m) \rfloor + 1}$.

With the notation above, each list of three numbers *uniquely* identifies a three-heap position, and vice versa. For our convenience and without any confusion, we also use $[0, b, c]$ to denote two-heap positions.

We will also abuse the definition of (special) Wythoff's sequence by replacing the requirement of $B_n = A_n + n$ to $B_{n+1} - A_{n+1} = B_n - A_n + 1$ when n is large enough, because we can obtain a Wythoff's sequence by chopping off a number of pairs from the sequence and reorganizing the indices.

The conjectures can now be rephrased as follows. For any given $m \geq 0$, $[A_n^m, A_n^m + B_n^m]$ is a special Wythoff's sequence. In other words, N_1^m, N_2^m and α^m exist.

Claim 3.1. *$T^m = \{b : \exists a < m,$ such that $[a, m-a, b] \in P^a\}$, thus it is finite.*

Proof: Denote the right-hand side of the equation as T_1. If $b \in T_1$, $[m, b, c]$, is an N-position for any $c \geq 0$, because we can simply remove an appropriate number of tokens from the third pile to create a P-position. Similarly, $[m, c, b-c]$ is also an N-position for any $c \leq b$, because we can remove tokens from the second pile. Hence $T_1 \subset T^m$. On the other hand, if $b \in T^m$, $[m, b, c]$ is an N-position for any $c \geq 0$. By the rules, to find the P-position corresponding to $[m, b, c]$ for each c, we can

I) remove b_1, b_2, b_3 tokens from the three piles correspondingly, providing $b_1 \oplus b_2 \oplus b_3 = 0$ and $b_1 + b_2 + b_3 > 0$;

II) remove a_1 tokens from the first pile where $0 < a_1 \leq m$;

III) remove b_1 tokens from the second pile where $0 < b_1 \leq b + m$;

IV) remove c_1 tokens from the third pile where $0 < c_1 \leq m + b + c$.

There are only finitely many choices involving moves I), II) and III), but infinitely many choices of c, so there must exist $0 \leq c' < b + m$, such that a P-position has m, c' and $b + m$ tokens in the piles. Since $b \in T^m$, $c' < m$. Thus $b \in T_1$ and $T^m \subset T_1$.

It is easy to see that $T^1 = \{1\}$ because of $[0, 1, 1]$; $T^2 = \emptyset$; and $T^3 = \{1, 2, 3\}$ because of $[1, 2, 1]$, $[0, 3, 2]$ and $[2, 1, 3]$.

We implement the following steps in order to prove the conjectures on the Wythoff's game for any specific m.

Claim 3.2. *We can represent positions in the three-heap Wythoff's game symbolically, and therefore can create a generating function to find P-positions.*

Proof: Given a P-position $[a, b, c]$, $a \leq m$, the following positions whose first piles have m tokens can reach this position with one move:

$$[m, b + b' - (m - a), c + ((m - a) \oplus b') - b'],\ b' \geq 0,\ if\ a < m \tag{3.1}$$

$$[m, a + a' - m, b + c + ((m - a - b) \oplus a') - a'],\ a' \geq m - a,\ if\ a + b < m \tag{3.2}$$

$$[m, a + a' - m, b + ((m - a - b - c) \oplus a') - a'],\ a' \geq m - a,\ if\ a + b + c < m \tag{3.3}$$

$$[m, b + b', c - b'],\ b' \leq c,\ if\ a = m \tag{3.4}$$

$$[m, b + c, b'],\ b' \geq 0,\ if\ a = m \tag{3.5}$$

$$[m, b, c + c'],\ c' \geq 0,\ if\ a = m \tag{3.6}$$

$$[m, b', c - b'],\ 0 \leq b' \leq c,\ if\ a + b = m \tag{3.7}$$

$$[m, c, c'],\ c' \geq 0,\ if\ a + b = m \tag{3.8}$$

$$[m, b', b + c],\ b' \geq 0,\ if\ a + b = m \tag{3.9}$$

$$[m, b', b],\ b' \geq 0,\ if\ a + b + c = m \tag{3.10}$$

The ten sets of positions above correspond to the following moves. Add tokens to the original position to:

- all three rows, with $m - a$ tokens to the first pile, b' second and $(m - a) \oplus b'$ third,
- all three rows, with a' first, $m - a - b$ second and $(m - a - b) \oplus a'$ third,
- all three rows, with a' first, $(m - a - b - c) \oplus a'$ second and $m - a - b - c$ third,
- the second row, but not enough to exceed the third,
- the second row, and exceeding the third,
- the third row only,
- the first row, but not enough to exceed the third,
- the first row, and exceeding the third,
- the first and the third rows,
- the first and second rows, and both exceeding the third.

Also in cases of 3.1, 3.2 and 3.3, we may need to increase the second and third numbers to avoid possible negative values: if $[m, b', c']$ is the resulting N-position and $c' < 0$, we will change it to $[m, b' + c', -c']$; and if b' or $b' + c'$ is less than 0, we simply replace it with 0.

Therefore, if $[a, b, c]$ is a P-position, the positions listed above are all the N-position deduced from it. So for each position (A^1, A_n^2, A_n^3) in the game, N or P, by fixing the first pile, we can use $x_1^{A_n^2} x_2^{A_n^3}$ to represent it symbolically. By observing that for any given b,

$(b \oplus c) - c$ is periodic as a function of c with period $p(b)$. we know that for each P-position $[a, b, c]$, all the N-positions deduced from it, possibly including $[a, b, c]$ itself, are:

$$\sum_{k=0}^{p(m-a)-1} \frac{x_1^b x_2^c}{1 - x_2^{((m-a)\oplus k)-k}}, \ if \ a < m \tag{3.11}$$

$$\sum_{k=0}^{p(m-a-b)-1} \frac{x_2^{b+c}}{1 - x_2^{((m-a-b)\oplus(m-a+k))-(m-a+k)}}, \ if \ a < m \tag{3.12}$$

$$\sum_{k=0}^{p(m-a-b-c)-1} \frac{x_2^{b+c}}{1 - x_2^{((m-a-b-c)\oplus(m-a+k))-(m-a+k)}}, \ if \ a < m \tag{3.13}$$

$$\sum_{k=0}^{c} (x_1^{b+k} x_2^{c-k}), \ if \ a = m \tag{3.14}$$

$$\frac{x_1^{b+c}}{1 - x_2}, \ if \ a = m \tag{3.15}$$

$$\frac{x_1^b x_2^c}{1 - x_2}, \ if \ a = m \tag{3.16}$$

$$\sum_{k=0}^{c} (x_1^k x_2^{c-k}), \ if \ a + b = m \tag{3.17}$$

$$\frac{x_1^c}{1 - x_2}, \ if \ a + b = m \tag{3.18}$$

$$\frac{x_2^{b+c}}{1 - x_1}, \ if \ a + b = m \tag{3.19}$$

$$\frac{x_2^b}{1 - x_1}, \ if \ a + b + c = m \tag{3.20}$$

Given a position $[a, b, c]$, let $N([a, b, c])$ be the set of all positions that can reach $[a, b, c]$ with one move, and denote by $f([a, b, c])$ the sum of the formal series 3.11–3.20, whenever applicable. Then f is the sum of the symbolic representations of $[a, b, c]$ and $N([a, b, c])$. We now define:

$$F_1(x_1, x_2) = \sum_{[a,b,c] \in P^a, \ all \ a<m} f([a, b, c]) \quad ,$$

$$F_2(x_1,x_2) = \sum_{[a,b,c]\in P^m} f([a,b,c]) \quad , \quad and$$

$$F(x_1,x_2) = \frac{1}{(1-x_1)(1-x_2)} - F_1(x_1,x_2) - F_2(x_1,x_2) \quad .$$

The sum for $F_2(x_1,x_2)$ is over the set of all known P-positions in P^m; and the sum for $F_1(x_1,x_2)$ is over the set of all P-positions whose first pile have less than m tokens with b and c large enough. We consider the Taylor expansion of $F(x_1,x_2)$ and find the lexicographically first monomial with strictly positive coefficient. The next P-position will be $[m,b,c]$, which always exists based on the rules of the game. In practice, we will use a faster approach to find A^m_n, namely $A^m_n = mex\{A^m_i, A^m_i + B^m_i : 0 \le i < n\}$, and still use the generating function to find B^m_i. The reason that we can use *mex* here is because of our assumption $A^m_{n-1} < A^m_n$, which indicates that any integer between the two must be in $\{A^m_i + B^m_i\}_{0\le i<n} \cup T$.

The game can also be visualized as follows. When the first pile has a fixed amount, m, of tokens, consider all the positions as points in the first quadrant with integral coordinates. For example, a position $[m,b,c]$ will be represented by the point (b,c) in our coordinate system. We call *instant winners* the positions at which a player can declare himself a winner immediately. In our game, they are the positions $[m,b,c]$ such that they can reach a certain $[a',b',c'] \in P^a$ with $a' < m$. Cross these points out of our coordinate system, and find the first point $[b,c]$ that has not been erased with the smallest possible x, and for that x, the smallest y coordinate. By the rules of the game, $[m,b,c]$ is a P-position. After finding each P-position$[m,b,c]$, we draw the following lines starting from (b,c): an upward vertical line, a leftward horizontal line, a 45° south-eastern slant line and an upward vertical line starting from the x-intercept of the slant line. Remove all the points on the lines, because they are the N-positions that can reach the newly found P-position with one move. Repeat the process to find the next P-position.

In Figure 1, $m = 1$; each (small) cross is an instant winner; and each "X" is a P-position.

Claim 3.3. *Given all* $[m, A^m_i, B^m_i] \in P^m$*, with* $i \le N$*, We can decide whether a given integer* $c < B^m_N$ *is in* S^m *by the following rules.*

- *if there exists* $[a,b,c] \in P^a$ *and* $a+b = m$*, then* $b+c \in S^m$*;*
- *if there exists* $[a,b,c] \in P^a$ *and* $a+b+c = m$*, then* $b \in S^m$*;*
- *if there exists* n *such that* $B_i \ne c$ *when* $i < n$*;* $B_n > c$*; and coeff*$(F_{2,n}(x_1), x_1, i) \le 0$ *with* $A_n \le i \le A_n + p(m)$*, then* $c \in S^m$*.*

Proof: Here we can see the advantage of symbolic over numeric computing, even though the latter would have been faster if we were *only* looking for the next P-positions.

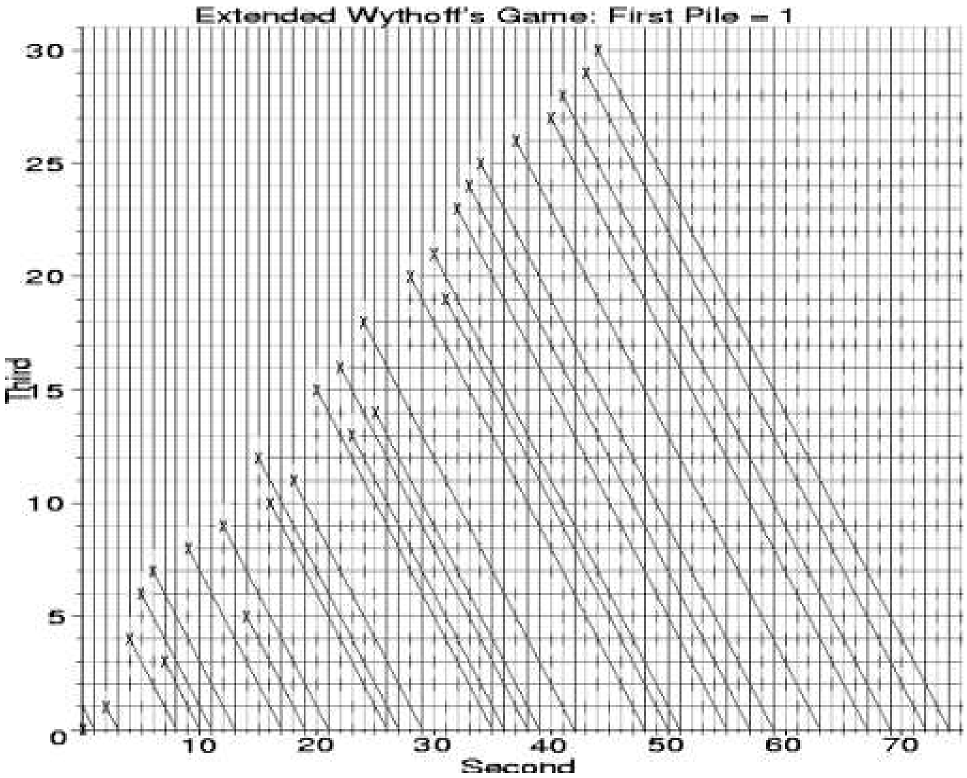

FIGURE 1. P-positions and instant winners when $m = 1$ and $n \leq 28$

Consider the generating function $F_1(x_1, x_2)$ generated by all the P-positions, whose first piles have less than m pieces, and their induced N-positions, namely, the sum of the formal power series 3.11, 3.12, 3.13, 3.17, 3.18, 3.19 and 3.20 over all the P-positions described above. Let $F_{1,n}(x_1, x_2)$ be the Taylor expansion of $F_1(x_1, x_2)$ of degree $\max\{A_i^m + B_i^m : i \leq n\} + p(m)$. Denote $\mathrm{coeff}(f(x), x, n)$ as the coefficient of x^n of the Taylor expansion of $f(x)$, and let $F_{2,n}(x_1)$ be $\mathrm{coeff}(F_{1,n}(x_1, x_2), x_2, c)$. So from the proof of the previous claim, $c \in S^m$ if we can show that there exists N such that $B_n^m \neq c$ when $A_n^m \leq N$, and $\mathrm{coeff}(F_{2,n}(x_1), n)$ are all positive when $n > N$.

The first two cases are obvious because we can remove the same number of tokens from two different piles, or symbolically we can use 3.19 and 3.20, and work with the coefficients of the corresponding formal series. In the third case, we only consider the moves that remove tokens from all three piles, or equivalently, cases 3.1, 3.2 and 3.3, which generate formal series 3.11, 3.12 and 3.13. By our notation, each monomial $x_1^a x_2^b$ with positive coefficient in the Taylor expansions of the fractional expressions represents either the P-position that generates the terms or an N-position $[m, a, b]$ deduced from the P-position. By the previously mentioned fact that $(a \oplus b) - b$ is periodic as a function of b, with period $p(a)$, which divides $p(m)$ if $a \leq m$. Therefore if such an n exists as described above, $\mathrm{coeff}(F_{2,n}(x_1), x_1, i) \geq 0$ for any

$i \geq A_n$, and this finishes the proof of our claim. We denote all such numbers as $S_1^m(n)$, which is a subset of S^m.

For example when $m = 1$, $2 \in S^1$ because $[0, 1, 1] \in P^0$; $17 \in S^1$ because $[1, 24, 18] \in P^1$ and checking the generating function at that point confirms the result. In fact, manual check indicates that when $n < 29$, there is no P-position of the form $[1, n, 17]$; when $n \geq 15$, $[1, 2n - 1, 17]$ are all N-positions because $[0, 29, 18] \in P^0$, and $[1, 2n, 17]$ are all N-positions because $[0, 25, 16] \in P^0$. Such manual checks become impractical as m grows larger. Interested readers can try to check another simple one: $22 \in S^1$.

In the language of instant winners, $c \in S_1^m(n)$ means the instant winners will fill the horizontal line $y = c$ for $x > A_n^m$, e.g. check $c = 2, 17$ in Figure 1.

Claim 3.4. *There exists an integer N such that when $n > N$, $A_n^m > \max\{T^i : i \leq m\}$ and $B_n^m > m + \max\{T^i : i \leq m\}$. If for a given $n > N$, $B = \max(\{B_i^m : i \leq n\})$, $mex(\{B_i^m : i \leq n\} \cup S_1^m(n)) = B + 1$, and $B + 1 - B_{n'}^i > m - a$ whenever $i + A_{n'}^i \leq m + A_{n+1}^m$ with $0 \leq i < m$, then $B_{n+1}^m = B + 1$.*

Proof: Since T^i, $i < m$, are all finite, there must exist an integer N as specified. If there is an $n > N$ as described in the assumption, let us consider the position $[m, A_{n+1}^m, B + 1]$. To prove it is a P-position, we only need to show that it cannot reach another P-position with one move. For any $[a, b, c] \in P^a$ with $a < m$ and $a + b \leq m + A_{n+1}^m$, the moves like 3.7, 3.8, 3.9 and 3.10 require $b \leq m$ and $c \in T^i$ for some $i < m$, so the moves can change the second number to at most $\max\{T^i : i \leq m\}$ or the third number to $m + \max\{T^i : i \leq m\}$. So these moves cannot affect $[m, A_{n+1}^m, B+1]$ being a P-position or N-position. Since $B+1-c > m-a$ and $(a_1 \oplus a_2) - a_2 \leq a_1$ for any a_1 and a_2, when the moves like 3.1, 3.2, 3.3 change the first and second numbers of $[a, b, c]$ to m and A_{n+1}^m, they can change the third number from c to at most $m - a + c$, i.e., the third pile has at most $m + A_{n+1}^m + m + c - a < m + A_{n+1}^m + B + 1$ tokens, which means these moves are all irrelevant too. Thus we have been freed from the possible moves that involve the first pile. On the other hand, all the moves that involve only the second and third piles, namely 3.4, 3.5 and 3.6, will not increase both of the second and the third numbers at the same time, so $B_{n+1}^m = mex(\{B_i^m : i \leq n\} \cup S^m) = B + 1$.

Using the visual interpretation of game, we can view the result as: $B_{n+1}^m = mex(\{B_i^m : i \leq n\} \cup S^m) + 1$, if the instant winners are not involved in the decision-making. For example, check the P-positions when $c \geq 23$ in Figure 1. This claim provides us a sufficient condition to verify *when* $B_{n+1}^m = B_n^m + 1$.

Claim 3.5. *For a given m, if N_1^i, N_2^i and N_3^i exist for $i < m$, the following conditions imply both conjectures for m: given an integer N as in Claim 3.4, if there exist $n_1 > n_2 > N$ such that*

- $A_{n_2+3}^m + B_{n_2+3}^m < A_{n_1}^m$*;*
- $B_{j+1}^m = B_j^m + 1$ *for* $n_2 \leq j \leq n_1$*;*

- $B^m_{n_1} = \max(\{B^m_i : i \leq n_1\}$;
- $mex(\{B^m_i : i \leq n_1\} \cup S^m_1(n_1)) = B^m_{n_1} + 1$;
- $\max(\alpha^m_j : n_2 \leq j \leq n_1) - \min(\alpha^m_j : n_2 \leq j \leq n_1) \leq 2$;
- $B^m_{n_1} > B^i_{N^i_3}$, $i < m$;

Furthermore if we denote $\alpha' = \lfloor(\max(\alpha^m : n_2 \leq j \leq n_1) - \min(\alpha^m : n_2 \leq j \leq n_1))/2\rfloor$ *and* $\epsilon^m_i = m + A^m_i - \lfloor B^m_i \phi\rfloor - \alpha'$, $i \geq 1$, *we also assume:*

- $\alpha^i - \alpha' \geq 4(m - i)$, $0 \leq i < m$;
- $\epsilon^m_j \epsilon^m_{j-1} \epsilon^m_{j-2} = 0$, $n_2 < j < n_1$.

Proof: Note that although there are eight conditions in the assumption, the first six are in fact necessary conditions for the conjectures.

We prove this Claim by induction and assume all the conditions are true for $n \geq n_1$, i.e., $mex(\{B^m_i : i \leq n\} \cup S^m_1(n)) = B^m_n + 1$, $B^m_n = B^m_{n-1} + 1$, and $|\epsilon_n| \leq 1$.

To prove $B^m_{n+1} = B^m_n + 1$, by Claim 3.4, we need to show that if $i + A^i_{n'} \leq m + A^m_{n+1}$ with $i < m$, then $B^m_n + 1 > B^i_{n'} + m - i$. Since $A^m_{n+1} \leq A^m_n + 2$, $\lfloor B^m_n\phi\rfloor + \alpha' + \epsilon^m_n + 2 - \lfloor B^i_{n'}\phi\rfloor - \alpha^i - \epsilon^i_{n'} \geq 0$, therefore $\lfloor B^m_n\phi\rfloor - \lfloor B^i_{n'}\phi\rfloor \geq \alpha^i - \alpha' - 4$; $(B^m_n - B^i_{n'})\phi + 1 > \alpha^i - \alpha' - 4$; $B^m_n + 1 - B^i_{n'} > (\alpha^i - \alpha' - 5)/\phi + 1$; so $B^m_n + 1 - B^i_{n'} \geq m - i$. The only time that the equal sign may hold is when $m = i + 1$, $\alpha^i - \alpha' = 4$, $\epsilon^i_{n'} = -1$, $\epsilon^m_n = 1$, and $B^m_n = B^i_{n'} = B$. By Lemma 2.4 and the assumption $B^i_{n'} = B^m_n > B^i_{N^i_3}$, $\epsilon^i_{n'-1} = \epsilon^m_{n-1} = 0$. Thus $A^m_n - A^m_{n-1} = \lfloor B\phi\rfloor - \lfloor(B-1)\phi\rfloor + 1$ and $A^i_{n'} - A^i_{n'-1} = \lfloor B\phi\rfloor - \lfloor(B-1)\phi\rfloor - 1$. If $\lfloor B\phi\rfloor - \lfloor(B-1)\phi\rfloor = 1$, $A^i_{n'} - A^i_{n'-1} = 0$; and if $\lfloor B\phi\rfloor - \lfloor(B-1)\phi\rfloor = 2$, $A^m_n - A^m_{n-1} = 3$. Neither of the two cases is possible. So $B^m_{n+1} = B^m_n + 1$ and $|\epsilon^m_{n+1}| \leq 1$ by Lemma 2.4. It is also obvious now that $S^m_1(n) = S^m_1(n+1)$, $B^m_{n+1} = \max(\{B^m_i : i \leq n+1\}$ and $mex(\{B^m_i : i \leq n+1\} \cup S^m_1(n+1)) = B^m_{n+1} + 1$, therefore we have completed the induction. In this case, $\alpha^m = \alpha'$ and $S^m = S^m_1(n_1)$.

Claim 3.6. *When* $m < 10$, $\{(A^m_n, A^m_n + B^m_n)\}_{n \geq 0}$ *are special Wythoff's sequences, and thus we have proved the conjectures.*

Proof: By using Claim 3.5, we only need to show that the two integers n_1 and n_2 do exist. Table 1 lists results for $m \leq 10$, in which we still use the same notations T^m, S^m, α^m, N^m_1 and N^m_2 as described at the beginning of the section. Complete results and the associated Maple package are available at
http://math.temple.edu/∼xysun/wythoff/wythoff.htm.

As we can see, the results for $m = 1$ are consistent with the ones predicted by Fraenkel [7], that also appear in Guy, R. J. Nowakowski [11], since the 21st and 28th P-positions are $[1, 32, 23]$ and $[1, 44, 30]$ respectively. (Note that our notation differs slightly from that of [7], so some of the signs are reversed).

m	T^m	S^m	α^m	N_1^m	N_2^m
0			0	1	1
1	1	2, 17, 22	-4	21	28
2		1, 5, 8, 24, 26, 32	-10	28	58
3	1, 2, 3	2, 3, 4, 5, 8, 10, 11, 12, 28, 41, 57	-16	48	73
4	3, 6	2, 5, 6, 7, 8, 9, 11, 12, 14, 17, 46, 48, 59	-20	126	208
5	4, 7, 10	1, 3, 5, 6, 8, 10, 11, 12, 15, 16, 17, 18, 19, 28, 56, 77, 83	-26	71	123
6	2, 3, 4, 6	1, 2, 3, 4, 5, 7, 8, 10, 11, 12, 13, 14, 15, 16, 18, 21, 44, 58, 95, 96, 132	-32	113	232
7	1, 4, 6, 7, 9	0, 1, 2, 4, 5, 6, 7, 8, 10, 12, 13, 14, 15, 17, 18, 19, 21, 22, 23, 24, 28, 30, 86, 88, 232, 251	-39	227	343
8	3, 5, 10, 13	1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 19, 20, 21, 22, 24, 26, 27, 28, 33, 34, 46, 155, 257, 390, 415	-46	388	648
9	6, 4, 11, 15	0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 24, 25, 26, 27, 30, 36, 37, 44, 48, 62, 254, 388, 421, 676	-52	645	645
10	6, 7, 8, 9, 13	0, 1, 2, 3, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 25, 27, 28, 29, 30, 31, 32, 34, 39, 50, 53, 103, 391, 424, 690	-56	656	656

Table 1. Results on Three-Heap Wythoff's Games

4. Epilogue

The method discussed here should be able to be extended to prove the conjectures for Wythoff's games with more than three heaps. A numerical method, instead of the symbolic one presented here, may also be developed to improve the performance, provided Claim 3.3 can be proved without using the generating functions. We hope the result presented here would be a stepping-stone for others to finally prove the conjectures, and better yet, to provide a *constructive* polynomial-time winning strategy for the game.

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