There are $\frac{1}{30}(\mathbf{r+1})(\mathbf{r+2})(\mathbf{2r+3})(\mathbf{r^2+3r+5})$ Ways For the Four Teams of a World Cup Group to Each Have r Goals For and r Goals Against [Thanks to the Soccer Analog of Prop. 4.6.19 of Richard Stanley's (Classic!) EC1]

By Shalosh B. EKHAD and Doron ZEILBERGER

Dedicated to Richard Peter Stanley who just turned "number of ways for a simple 1D Drunkard to return home after 8 steps"-years-old

Proof of the Statement in the Title: Calling this quantity $S_4(r)$, and using the case $n=4$ of the Soccer analog of Prop. 4.6.19 of [EC1] (see Comment 1 below), we see that

- $S_4(r)$ is a polynomial of degree 5 .
- $S_4(-1)=0$, $S_4(-2)=0$.
- $S_4(-3-r) = -S_4(r)$.

Hence, it suffices to check the statement at the **two** values $r=0$ and $r=1$. Obviously $S_4(0)=1$ (there is just one all-zero 4×4 matrix), and almost obviously, $S_4(1)=9$ (the number of derangements of length 4 equals 9). QED!

Comments

1. The same method of proof that guru Richard Stanley used to prove Prop. 4.6.19 of [EC1] (first edition; it is Prop. 4.6.2 in the second edition.) [the first edition is missing $(-1)^{n-1}$ in front of $H_n(r)$] yields:

The Soccer analog of Prop. 4.6.19 of EC1

Let $S_n(r)$ be the number of ways n Soccer teams, playing a round-robin tournament, each scored a total of r "Goals For" (GF), and a total of r "Goals Against" (GA), (in other words the number of $n\times n$ magic squares with line-sums r *and* whose diagonal entries are all 0 (no team plays against itself!)).

For each fixed integer $n\geq 3$, the function $S_n(r)$ is a polynomial in r of degree n^2-3n+1. Since it is a polynomial, it can be evaluated at *any* (not necessarily positive) integer, and we have

$$S_n(-1)=S_n(-2)=\ldots=S_n(-n+2)=0 \quad ,$$

$$S_n(-(n-1)-r)=-S_n(r) \quad .$$

2. We got the idea for this tribute to our beloved guru when we attended the Stanley@70 conference, held at MIT, June 23-27, 2014, at the same time as the preliminary Group stage of the World Cup 2014.

3. While the proof of the statement of the title did not require any computer, the analogous result for $n = 5$ is already beyond the scope of mere humans (or the human would have to be very stupid to spend time on it). We have

$$S_5(r) = \frac{1}{241920} \, (r+1)\,(r+2)\,(r+3) \cdot$$

$$\left(43\, r^8 + 688\, r^7 + 4934\, r^6 + 20680\, r^5 + 55907\, r^4 + 101272\, r^3 + 123436\, r^2 + 96240\, r + 40320\right) \quad .$$

Of course $S_3(r) = r + 1$ (why?).

To see $S_6(r)$, please go to: `http://www.math.rutgers.edu/~zeilberg/tokhniot/oGOALS1` .

One can get this, and (potentially!) infinitely more results, using the Maple package **GOALS** available directly from: `http://www.math.rutgers.edu/~zeilberg/tokhniot/GOALS` , or via the front of this article: `http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/worldcup.html` .

In particular, procedure `MagicPolAG(n,r,Sr,Sc,B,A)` can find polynomial expressions for the number of $n \times n$ matrices with non-negative integer entries, whose row-sums are

$$(r + Sr[1], \dots, r + Sr[n]) \quad ,$$

and whose column sums are

$$(r + Sc[1], \dots, r + Sc[n]) \quad ,$$

for *any* fixed numerical vectors `Sr`, `Sc`, of length n (of course they have to add-up to the same number), and any assignment A where some entries are fixed beforehand, where B denotes "wild-card". See the on-line help there. See the above-mentioned front of this article for numerous sample input and output files.

4. The sequence for $S_4(r)$ is already in Neil Sloane's OEIS, (see [OEIS1]), but for **different** reasons! Can you find a bijection? On the other hand, the sequence for $S_5(r)$ is not (yet!) there:

$$1, 44, 870, 9480, 68290, 365936, 1573374, 5709120, 18107760, 51488800, 133748186, 321979164, \dots \quad ,$$

but we are sure that very soon it will!

5. Another, even more useful, Maple package accompanying this article is `WorldCup`, one of whose many procedures is '`Ptor`', that finds all the possible scenarios that lead to a given *score board*, consisting of the "Goals For", "Goals Against", and "PTS", vectors. (Recall that a win yields 3 points, a draw, 1 point, and a loss, 0 points.).

The number of possible scenarios for groups A-H were as follows

2014: $8, 32, 7, 3, 13, 3, 12, 3$.

2010: $2, 3, 2, 2, 3, 3, 1, 2$ (in particular, the score-board for Group G *uniquely* determined the individual game scores).

2006: $11, 1, 5, 3, 2, 6, 1, 8$ (in particular, the score-boards for Groups B and G *uniquely* determined the individual game scores).

2002: $1, 12, 5, 5, 1, 2, 3, 1$ (in particular, the score-boards for Groups A, E, H *uniquely* determined the individual game scores).

1998: $9, 6, 4, 3, 3, 3, 1, 1$ (in particular, the score-boards for Groups G, H *uniquely* determined the individual game scores).

6. The Israeli daily *Yedioth Ahronoth* published, at the start of the 2014 World Cup, a "soccer puzzle" where the above-mentioned vectors, GF, GA, and PTS, are given, and the solver has to reconstruct the scores of the individual matches. The above-mentioned Maple package `WorldCup` has a procedure, `Khida`, that makes up such puzzles, and another procedure, `Sefer1`, that creates challenging puzzle books. For some samples, see

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oWorldCupi` , for $1 \leq i \leq 4$. Enjoy!

7. Happy birthday Richard, and keep up the good work! (and continue to practice what Phil Hanlon calls the "the three H's").

References

[OEIS1] Neil J.A. Sloane, *Sequence A061927*, `http://oeis.org/A061927`.

[EC1] Richard P. Stanley, "*Enumerative Combinatorics, volume I*", first edition: Wadsworth & Brooks/Cole, 1986; second edition: Cambridge University Press, 2011.
Available on-line (viewed July 7, 2014): `http://math.mit.edu/~rstan/ec/ec1.pdf` .

Shalosh B. Ekhad, c/o D. Zeilberger, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.

Doron Zeilberger, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.
url: `http://www.math.rutgers.edu/~zeilberg/` .
Email: `zeilberg at math dot rutgers dot edu` .

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