# SOME NICE SUMS ARE ALMOST AS NICE IF YOU TURN THEM UPSIDE DOWN

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ABSTRACT. We represent the sums $\sum_{k=0}^{n-1} \binom{n}{k}^{-2}$ , $\sum_{k=0}^{m} \binom{m}{k}^{-1} \binom{a}{n-k}^{-1}$, $\sum_{k=0}^{n-1} \frac{q^{-k(k-1)}}{\left[{n \atop k}\right]_q}$, and the sum of the reciprocals of the summands in Dixon's identity, each as a product of an *indefinite hypergeometric sum* times a (closed form) *hypergeometric sequence* .

In [3], Rockett proved the formula

$$\sum_{k=0}^{n} \binom{n}{k}^{-1} = \frac{n+1}{2^n} \sum_{j=0}^{n} \frac{2^j}{j+1} \ , \qquad (Rockett)$$

for all nonnegative integers $n$. Since then, several authors (e.g. [1][4][5]) used different techniques to re-derive (*Rockett*) and considered analogous, more complicated, sums.

As we all know,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \ , \qquad (Binomial)$$

is the *simplest* example of a **definite** binomial coefficient sum that evaluates in *closed-form*. Such sums have the form

$$\sum_{k=0}^{n} F(n,k) = A(n) \ ,$$

where $A(n)$ is *hypergeometric* (i.e. $A(n+1)/A(n)$ is a rational function of $n$), and $F(n,k)$ is *bi-hypergeometric* (i.e $F(n+1,k)/F(n,k)$ and $F(n,k+1)/F(n,k)$ are both rational functions of $n$ and $k$). Other famous examples, are

$$\sum_{k=0}^{n} \binom{n}{k}\binom{a}{k} = \binom{n+a}{a} \ , \qquad (Chu-Vandermonde)$$

and

$$\sum_{k=-n}^{n} (-1)^k \binom{a+n}{n+k}\binom{b+n}{b+k}\binom{a+b}{a+k} = \frac{(a+b+n)!}{a!b!n!} \ . \qquad (Dixon)$$

The *best* thing that can happen to a binomial coefficients sum is to be evaluable in *closed form*, i.e. as a hypergeometric sequence. The *next best* thing is to be "almost closed-form", i.e. of the form

$$ClosedForm(n) \cdot \left( \sum_{j=1}^{n} h(j) \right) \quad ,$$

where $h(j)$ is hypergeometric in the *single* variable $j$.

Inspired by (*Rockett*), that has this "almost perfect" form, and that came out from considering the sum of the reciprocals of the summands of the *simplest* binomial coefficient sum known, we searched for other cases where one starts from a well-known binomial coefficients sum that evaluates in closed form, and looks at what happens if one considers the sum of the reciprocals of the summands. To our great surprise and delight, it worked for the Chu-Vandermonde summand and for the Dixon summand (see below). To our disappointment, it didn't work for the so-called Pfaff-Saalschutz identity (see, e.g. [2]).
We use the WZ method([2][7][8]), and the reader is assumed to be familiar with it. In particular we used Zeilberger's Maple packages `EKHAD` (procedure zeillim) and `qEKHAD` accompanying [2], available from

`http://www.math.rutgers.edu/~zeilberg/tokhniot/[EKHAD,qEKHAD]` .

In most applications of the WZ method, the summation limits are *natural*, i.e. the summand is well-defined and vanishes at $k = -1$ and $k = n + 1$ (or in the case of (*Dixon*), at $k = -n - 1$ and $k = n + 1$) so one can get a *homogeneous* first-order linear recurrence for the definite sum, that leads to a closed-form solution. In the present cases, this is no longer the case, and we only get *inhomogeneous* first-order linear recurrences, that lead to the above-mentioned "almost-perfect" kind of solutions.
Consider the sum

$$f(n) := \sum_{k=\alpha}^{n-\beta} F(n,k) \ ,$$

where $\alpha$ and $\beta$ are constants and $[\alpha, n - \beta]$ is properly contained in $[0, n]$, the support of $F(n,k)$. Assume also that there exists a function $G(n,k)$, (the so-called certificate) such that

$$a(n)F(n+1,k) + b(n)F(n,k) = G(n,k+1) - G(n,k) \ . \qquad (WZeqn)$$

Take $\alpha = 0$ and $\beta = 1$. Then, if we add both sides of ($WZeqn$) from $k = 0$ to $k = n$ and rewrite the result we get a nonhomogeneous recurrence relation satisfied by $f(n)$:

$$a(n)f(n+1) + b(n)f(n) = G(n,n) - G(n,0) + a(n)F(n+1,n) \ . \quad (nonhomorec)$$

Finally, we solve (*nonhomorec*) and get the following representation of $f(n)$ as an indefinite sum:

$$f(n) = g(n) \sum_{i=0}^{n-1} \frac{C(i)}{g(i)a(i)} \ ,$$

where

$$C(i) = G(i,i) - G(i,0) + a(i)F(i+1,i) \ .$$

(i.e. the right side of (*nonhomorec*) with $n$ replaced by $i$), and $g(n)$ is a solution to the associated homogeneous equation $a(n)f(n+1)+b(n)f(n)=0$.

We note that in case $\alpha \neq 0$ and $\beta \neq 1$, the left-hand side of (*nonhomrec*) remains the same. The only change is on the right-hand side.

**Theorem 1 [Upside Down Binomials Squared]**:

$$\sum_{k=0}^{n-1} \binom{n}{k}^{-2} = \frac{(n+1)!(n+1)^2}{(n+\frac{3}{2})!4^n} \sum_{j=0}^{n-1} \frac{(3j^3+12j^2+18j+10)4^j(j+3/2)!}{(j+1)^2(j+1)!(j+2)^3} .$$

(Here $n! := \Gamma(n+1)$).
**Proof**: We construct the function

$$G(n,k) = \frac{(2k-3n-6)(k-n-1)^2}{\binom{n}{k}^2}$$

such that $(4n+10)(n+1)^2F(n+1,k) - (n+2)^2F(n,k) = G(n,k+1) - G(n,k)$, where $F(n,k)$ is the summand on the left-hand side sum.

Next add both sides from $k=0$ to $k=n$ and rearrange to get the nonhomogeneous recurrence relation satisfied by the sum on the left-hand side, $S_n$ :

$$(4n+10)(n+1)^2S_{n+1} - (n+2)^2S_n = 3n^3+12n^2+18n+10 \ .$$

Finally the theorem follows by solving this recurrence with the initial condition $S_1 = 1$.

Next we give q-analog of (*Rockett*). Let $(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$.

**Theorem 2 [Upside Down q-binomial]** :

$$\sum_{k=0}^{n-1} \frac{q^{-k(k-1)/2}}{\begin{bmatrix} n \\ k \end{bmatrix}_q} = \frac{(q^2;q)_n}{(q^2;q^2)_n} \sum_{i=0}^{n-1} \frac{C(i)(q^2;q^2)_i}{(q^2;q)_i(q^{i+2}-1)} ,$$

where

$$C(i) = \frac{q^{i+1}+q^{3(i+1)}+q^{-i(i-1)/2}+q^{-(i+1)(i-4)/2}-q^{(2+3i-i^2)/2}-q^{-(i+1)(i-2)/2}-2q^{2(i+1)}}{q^{i+1}-1} .$$

**Proof**: We construct the function

$$G(n,k) = \frac{q^{n+1}(1-q^{n-k+1})q^{-k(k-1)/2}}{\begin{bmatrix} n \\ k \end{bmatrix}_q}$$

such that $(q^{n+1}-1)(q^{n+1}+1)F(n+1,k) - (q^{n+2}-1)F(n,k) = G(n,k+1) - G(n,k)$, where $F(n,k)$ is the summand on the left-hand side.

Add both sides from $k=0$ to $k=n$ and rearrange to get the nonhomogeneous recurrence relation satisfied by the sum on the left-hand side, $S_n$ :

$$(q^{n+1}-1)(q^{n+1}+1)S_{n+1}-(q^{n+2}-1)S_n = C(n) \ .$$

Finally the theorem follows by solving this recurrence with the initial condition $S_1 = 1$.

Next we consider the reciprocal of the summand in the Chu-Vandermonde and the Dixon classical identities.

**Theorem 3 [Upside Down Chu-Vandermonde]**

$$\sum_{k=0}^{m-1} \binom{m}{k}^{-1} \binom{a}{n-k}^{-1} = g(m) \sum_{i=0}^{m-1} \frac{C(i)}{g(i)(i+2)(a+i-n+2)}$$

where

$$g(m) = \frac{(a+m-n+1)!(a+4)!(m+1)}{2(a+m+3)!(a-n+2)!}$$

and

$$C(i) = (n+i+in+1)\binom{a}{n}^{-1} + (2i+a-n+3)\binom{a}{n-i}^{-1} \ .$$

**Proof**: The proof follows similarly from the recurrence relation

$$(a+m+4)(m+1)S_{m+1} - (m+2)(a+m-n+2)S_m = C(m) \ .$$

where $S_m$ is the sum on left-hand side.

**Theorem 4 [Upside Down Dixon]**

$$\sum_{k=0}^{n-1} (-1)^k \binom{n+b}{n+k}^{-1} \binom{n+c}{c+k}^{-1} \binom{b+c}{b+k}^{-1} = g(n) \sum_{i=0}^{n-1} \frac{C(i)}{g(i)(2i+2)(c+i+2)(b+i+2)}$$

where

$$g(n) = \frac{(b+n+1)(c+n+1)(b+c+2)!n!}{(b+1)(c+1)(b+c+n+2)!}$$

and

$C(i) = (-1)^i(3bi+3ci+bc+3b+5i^2+12i+3c+7)\binom{i+b}{2i}^{-1}\binom{b+c}{b+i}^{-1} - (b+1)(c+1)(i+1)\binom{i+b}{i}^{-1}\binom{i+c}{c}^{-1}\binom{b+c}{b}^{-1}$ .

**Proof**: The proof follows similarly from the recurrence relation

$$2(c+n+1)(b+n+1)(b+c+n+3)S_{n+1}-2(n+1)(c+n+2)(b+n+2)S_n=C(n)\ .$$

where $S_n$ is the sum on left-hand side.

**Remarks** a) From the recurrence in the proof of theorem 1,

$$2S_n=\frac{(n+1)^3}{2n^3+3n^2}S_{n-1}+\frac{3n^3+3n^2+3n+1}{2n^3+3n^2}$$

that implies

$$\lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}^{-2}=2\ .$$

b) Identity (*Rockett*) is a special case of the following identity with $x=y=1$ in

$$\sum_{k=0}^{n}\binom{n}{k}^{-1}x^k y^{n-k}=x^n+\left(\frac{xy}{x+y}\right)^n(n+1)\sum_{j=0}^{n-1}\frac{((j+1)y^{j+2}+yx^{j+1})(x+y)^j}{(xy)^{j+1}(j+1)(j+2)}$$

which follows from the recurrence [$S_n$ the left-hand side sum]:

$$(x+y)(n+1)S_{n+1}-(n+2)xyS_n=(n+1)(x^{n+2}+y^{n+2})\ .$$

[Added July 17, 2015: the previous version had the wrong right side. We thank Dennis Bernstein for spotting this error.]

c) T. Mansour [1] derived the following representation for the reciprocals of binomial coefficients to any power $m$ as:

$$\sum_{k=0}^{n-1}\binom{n}{k}^{-m}=(n+1)^m\sum_{k=0}^{n}\left[\sum_{j=0}^{k}\frac{(-1)^j}{n-k+1+j}\binom{k}{j}\right]^m\ .$$

One of the referees kindly pointed out that (b) and (c) can also be deduced from a general identity derived in [6]. We thank them for this remark and for numerous corrections and improvements.

## References

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