A SHORT PROOF OF JACOBI'S FORMULA FOR THE NUMBER OF REPRESENTATIONS OF AN INTEGER AS A SUM OF FOUR SQUARES

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Diophantus probably knew, and Lagrange[L] proved, that every positive integer can be written as a sum of four perfect squares. Jacobi[J] proved the stronger result that the number of ways in which a positive integer can be so written[3] equals 8 times the sum of its divisors that are not multiples of 4. Here we give a short new proof that only uses high school algebra, and is completely *from scratch.* All infinite series and products that appear are to be taken in the entirely elementary sense of formal power series.

The problem of representing integers as sums of squares has drawn the attention of many great mathematicians, and we encourage the reader to look up Grosswald's[G] erudite masterpiece on this subject.

The crucial part of our proof is played by two simple identities, that we state as one Lemma.

Lemma : Let

$$H_n = H_n(q) = \frac{1+q}{1-q}\frac{1+q^2}{1-q^2}\cdots\frac{1+q^n}{1-q^n}. \text{ For all integers } n \geq 0.$$

$$\sum_{k=-n}^{n} \frac{4(-q)^k}{(1+q^k)^2} H_n^2 H_{n+k} H_{n-k} = 1, \tag{a}$$

$$\sum_{k=0}^{n} \frac{2(-q^{n+1})^k}{1+q^k}\frac{H_k}{H_n} = \sum_{k=-n}^{n} (-q)^{k^2}. \tag{b}$$

Proof: Let $L_1(n)$ and $L_2(n)$ be the left sides of (a) and (b) respectively, and let $F_1(n,k)$, and $F_2(n,k)$ be the respective summands. Since both (a) and (b) obviously hold for $n=0$, it suffices to prove that for every $n \geq 0$, $L_1(n+1) - L_1(n) = 0$, and $L_2(n+1) - L_2(n) = 2(-q)^{(n+1)^2}$. To this end, we construct

$$G_1(n,k) := \frac{q^{n-k+1}(1+q^{2n+2})(1+q^k)^2(1+q^{n+k+1})}{(1-q^{n+1})^3(1-q^{n+k+1})(1+q^{n+1})} F_1(n,k), \; G_2(n,k) := \frac{(-q^{n+1})(1+q^k)}{1+q^{n+1}} F_2(n,k),$$

with the motive that

$$F_1(n+1,k) - F_1(n,k) = G_1(n,k) - G_1(n,k-1), \; F_2(n+1,k) - F_2(n,k) = G_2(n,k) - G_2(n,k-1), \quad (1)$$

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[3] Precisely: the number of vectors (*not sets*) (x_1, x_2, x_3, x_4), where the components are (positive, negative, or zero) integers, such that $x_1^2 + x_2^2 + x_3^2 + x_4^2 = n$.

which immediately imply them, by telescoping, upon summing from $k = -n-1$ to $k = n+1$, and from $k = 0$ to $k = n+1$ respectively. The two identities of (1) are purely routine, since dividing through by $F_1(n,k)$ and $F_2(n,k)$ respectively, lead to routinely-verifiable high-school-algebra identities.

Dividing both sides of (a) by H_n^4 and letting $n \to \infty$ in (a) and (b) gives

$$1 + 8\sum_{k=1}^{\infty} \frac{(-q)^k}{(1+q^k)^2} = H_\infty^{-4}, \qquad (a')$$

$$\sum_{k=-\infty}^{\infty} (-q)^{k^2} = H_\infty^{-1}. \qquad (b')$$

Combining (a') and (b'), yields, after changing $q \to -q$,

$$\left(\sum_{k=-\infty}^{\infty} q^{k^2}\right)^4 = 1 + 8\sum_{k=1}^{\infty} \frac{q^k}{(1+(-q)^k)^2}\ . \qquad (2)$$

The coefficient of a typical term q^n on the left of (2) is the number of ways of writing n as a sum of four squares. It remains to show that the coefficient of q^n in the sum on the right of (2) equals the sum of the divisors of n that are not multiples of 4.

Using the power-series expansion $z/(1+z)^2 = \sum_{r=1}^{\infty}(-1)^{(r+1)} r z^r$, with $z = (-q)^k$, and collecting like powers, the sum on the right side may be rewritten

$$\sum_{k=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{(k+1)(r+1)}\, r\, q^{kr} = \sum_{n=1}^{\infty} q^n \,[\sum_{r|n}(-1)^{(r+1)(n/r+1)} r\,].$$

The coefficient of q^n above is a weighed sum of divisors r of n, where the coefficient of r is $-1 = +1-2$ if both r and n/r are even and $+1$ otherwise, so the coefficient of q^n is

$$\sum_{r|n} r - \sum_{\substack{r|n \\ r,n/r\ even}} 2r = \sum_{d|n} d - \sum_{\substack{d|n \\ 4|d}} d = \sum_{\substack{d|n \\ 4\not| d}} d\,.$$

The finitary identities (a) and (b) combine to yield a single finitary identity

$$\left(\sum_{k=0}^{n} \frac{2(-q^{n+1})^k}{1+q^k} H_k\right)^4 \sum_{k=-n}^{n} \frac{4(-q)^k}{(1+q^k)^2}\frac{H_{n+k}}{H_n}\frac{H_{n-k}}{H_n} = \left(\sum_{k=-n}^{n} (-q)^{k^2}\right)^4\,, \qquad (3)$$

which also immediately implies Jacobi's theorem, by taking it "mod q^n" for any desired n. Identity (3) makes it transparent that our proof only uses the potential infinity, not the ultimate one.

The identities of the Lemma are examples of q-binomial coefficient identities, a.k.a terminating basic hypergeometric series identities. The proof of such identities is now completely routine[WZ][Z]. The proof of the Lemma given here used the algorithm of [Z]. Further applications of basic hypergeometric series to number theory can be found in [A1]. An excellent modern reference to basic hypergeometric series is [GR].

We conclude with some comments addressed mainly to the cognoscenti. Identities (a) and (b) are special cases of classical identities: (a) is a special case of Jackson's theorem [GR, p. 35, eq. (2.6.2)], and (b) is a special case of Watson's q-analog of Whipple's theorem ([GR, p.35, eq. (2.5.1)], see also [A2, p. 118, eq. (4.3)].) The discovery of (b) was motivated by [S1] and [S2].

We see fairly clearly how to do the 2-square theorem (a different instance of Jackson's theorem replaces (a)); however the theorems for 6 and 8 squares apparently require (using this approach) some instance of the ${}_6\Psi_6$ summation theorem [GR, p. 128, (5.3.1)] (see [A1, pp. 461-465] for details). Since we do not know a finitary analog of the ${}_6\Psi_6$ summation, the question of a similar result for 6 and 8 squares is of interest.

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