A Treatise on Sucker's Bets

By Shalosh B. EKHAD and Doron ZEILBERGER

Preface

Consider a gambling game where each player has a die marked with dots (not necessarily the usual way). For each die, each face is equally likely to show up, in other words the dice are *fair*. Each player rolls his die simultaneously, and whoever has more dots on his landed face wins the round.

In 1970, Statistics giant, Bradley Efron, amazed the world ([G], ch. 22; [W]) by coming up with a set of four dice, let's call them A,B,C,D, whose faces are marked as follows

$$A = [1, 1, 5, 5, 5, 5] \quad , \quad B = [4, 4, 4, 4, 4, 4] \quad , \quad C = [3, 3, 3, 3, 7, 7] \quad , \quad D = [2, 2, 2, 6, 6, 6] \quad .$$

It turns out that

- die A beats die B (in the sense that A's chance of winning exceeds B's chance of winning) ;
- die B beats die C ;
- die C beats die D ;

but, **surprise surprise**,

- die D beats die A!

This was an amazing demonstration that "being more likely to win" is **not** a transitive relation. But that was only one example, and of course, instead of dice, we can use decks of cards, that Martin Gardner ([G], ch. 23) called **sucker's bets**.

Gardner gave an example of a set of three decks of three cards each, all marked with different numbers, that is derived from the rows of a three by three magic square

$$A = [1, 6, 8] \quad , \quad B = [3, 5, 7] \quad , \quad C = [2, 4, 9] \quad .$$

Can you find **all** such examples, with a specified number of decks, and deck-sizes? If you have a computer algebra system (in our case Maple), you sure can!

Not only that, we can figure out how likely such sucker's bets are, and derive, **fully automatically**, statistical information!

Why are they called Sucker's Bets

The *sucker* tacitly assumes that 'being more likely to win' is a transitive relationship, hence he or she would not object to the privilege of being **first** to pick which deck to play with. Then the *hustler* can always pick a better deck.

• If the sucker picks A, then the hustler will pick C ;

• If the sucker picks B, then the hustler will pick A ;

• If the sucker picks C, then the hustler will pick B .

Let's first convince ourselves that A is better than B, B is better than C, and C is better than A.

A **versus** B

Out of the **nine** (equally likely!) possibilities $[a, b]$

$$\{[1,3],[1,5],[1,7],[6,3],[6,5],[6,7],[8,3],[8,4],[8,7]\} \quad ,$$

A wins in 5 of them, namely $\{[6,3],[6,5],[8,3],[8,4],[8,7]\}$, while B wins in 4 of them, namely $\{[1,3],[1,5],[1,7],[6,7]\}$.

B **versus** C

Out of the **nine** (equally likely!) possibilities $[b, c]$

$$\{[3,2],[3,4],[3,9],[5,2],[5,4],[5,9],[7,2],[7,4],[7,9]\} \quad ,$$

B wins in 5 of them, namely $\{[3,2],[5,2],[5,4],[7,2],[7,4]\}$, while C wins in 4 of them, namely $\{[3,4],[3,9],[5,9],[7,9]\}$.

C **versus** A

Out of the **nine** (equally likely!) possibilities $[c, a]$

$$\{[2,1],[2,6],[2,8],[4,1],[4,6],[4,8],[9,1],[9,6],[9,8]\} \quad ,$$

C wins in 5 of them, namely $\{[2,1],[4,1],[9,1],[9,6],[9,8]\}$, while A wins in 4 of them, namely $\{[2,6],[2,8],[4,6],[4,8]\}$.

Other Examples of Surprising Non-Transitive Relations

Of course, non-transitivity is nothing new! An ancient example is *love*. Joe loves his wife, Joe's wife loves her mother, but Joe **hates** his mother-in-law.

Another example is the relation *being afraid of*. Indeed

- My dog is afraid of me ;
- My cat is afraid of my dog ;
- The mouse is afraid of my cat ;
- My wife is afraid of the mouse ;
- I am afraid of my wife.

Another, more "serious" example is in voting, where famously the 18th-century French polymath,

Marie Jean Antoine Nicolas de Caritat, Marquis de Condorcet,

came up with his famous **paradox**, also mentioned by Gardner([G], ch. 23), that inspired Kenneth Arrow's *impossibility theorem* (that earned him a Nobel!).

Another example is the intriguing **Penny ante** ([P], see also [G], ch. 23, and discussion in [NZ]).

Mapping Sucker's Bets to Words

First let's consider the case where there are k decks, where deck 1 has a_1 cards, deck 2 has a_2 cards, $\dots$, deck k has a_k cards, and let $N = a_1 + \dots + a_k$ be the total number of cards participating. Let's first treat the case where all the denominations of these N cards are different, so without loss of generality, we can make them $\{1, 2, \dots, N\}$. There are $(a_1 + \dots + a_k)!/(a_1! \cdots a_k!)$ ways of assigning the cards to the various decks, and there is an obvious bijection between such decks and words w in the alphabet $\{1, 2, \dots, k\}$ with a_1 occurrences of 1, a_2 occurrences of 2, $\dots$ a_k occurrences of k, where $w_i = j$ means that we put the card with the denomination i into the j-th deck.

For pedagogical clarity, until further notice, let's take $k = 3$.

For example, the above-mentioned set of three decks, each with three cards

$$A = [1, 6, 8] \quad , \quad B = [3, 5, 7] \quad , \quad C = [2, 4, 9] \quad ,$$

corresponds to the 'word'

$$ACBCBABAC \quad ,$$

and replacing A, B, C by $1, 2, 3$, respectively, we get the 'word'

$$132321213 \quad .$$

Which Words Correspond to Sucker's Bets?

Let's consider the "magic" deck above, whose 'word' turned out to be 132321213. The reason that it corresponds to a sucker's bet is that

• The number of times letter '1' is to the left of letter '2' (not necessarily immediately before) is less than the number of times that a '2' is ahead of a '1'.

(Indeed there are three '2's after the '1' at the first place, and one '2' after the '1' at the sixth place, totaling four occurrences of '1' before '2', while there are two '1's that occur after the '2' at the third place, two '1's that occur after the '2' at the fifth place, and one '1' that occurs after the '2' at the seventh place.)

• The number of times letter '2' is to the left of letter '3' (not necessarily immediately before) is less than the number of times that a '3' is ahead of a '2'. (Check!)

• The number of times letter '3' is to the left of letter '1' (not necessarily immediately before) is less than the number of times that a '1' is ahead of a '3'. (Check!)

This leads us to introduce three **word statistics**, for any word in the alphabet $\{1,2,3\}$ (below $|S|$ means, as usual, the number of elements in the set S).

$$s_1(w) := |\{(i,j)\,|\, i<j, w_i = 2 \quad AND \quad w_j = 1\}| - |\{(i,j)\,|\, i<j, w_i = 1 \quad AND \quad w_j = 2\}| \quad ,$$

$$s_2(w) := |\{(i,j)\,|\, i<j, w_i = 3 \quad AND \quad w_j = 2\}| - |\{(i,j)\,|\, i<j, w_i = 2 \quad AND \quad w_j = 3\}| \quad ,$$

$$s_3(w) := |\{(i,j)\,|\, i<j, w_i = 1 \quad AND \quad w_j = 3\}| - |\{(i,j)\,|\, i<j, w_i = 3 \quad AND \quad w_j = 1\}| \quad .$$

So for the above-mentioned $w = 132321213$, $s_1(w) = s_2(w) = s_3(w) = 1$.

This leads us to a **characterization** of words that correspond to sucker's bets.

Proposition: A word w in the alphabet $\{1,2,3\}$ corresponds to a sucker's bets (where the first deck has denominations indicating the locations of the letter 1, the second deck has denominations indicating the locations of the letter 2, and the third deck has denominations indicating the locations of the letter 3) **if and only if**

$$s_1(w) > 0 \quad , \quad s_2(w) > 0 \quad , \quad s_3(w) > 0 \quad . \qquad (SBC)$$

Counting Sucker's Bets

Given three positive integers a_1, a_2, a_3, out of the $(a_1+a_2+a_3)!/(a_1!a_2!a_3!)$ words w in the alphabet $\{1,2,3\}$ with a_1 1s, a_2 2s, a_3 3, how many have the property (SBC)?

In other words:

How many three-deck sets of cards form a sucker's bet where the first deck has a_1 cards, the second deck has a_2 cards, and the third deck has a_3 cards, and all denominations are different and are drawn from $\{1, 2, \ldots, a_1 + a_2 + a_3\}$?

The naive way would be to actually examine all the $(a_1 + a_2 + a_3)!/(a_1!a_2!a_3!)$ possible words, compute $s_1(w), s_2(w), s_3(w)$, and count those for which condition (SBC) holds. But there is a better way!

Let q_1, q_2, q_3 be three (commuting) *indeterminates*, and define the **weight** of a word w by

$$weight(w) := q_1^{s_1(w)} q_2^{s_2(w)} q_3^{s_3(w)} \quad . \qquad (Weight)$$

Let $\mathcal{W}(a_1, a_2, a_3)$ be the set of words in $\{1, 2, 3\}$ with a_1 1s,a_2 2s, and a_3 3s.

Now define the *weight enumerator* (aka *generating function*)

$$F(a_1, a_2, a_3)(q_1, q_2, q_3) := \sum_{w \in \mathcal{W}(a_1,a_2,a_3)} weight(w) = \sum_{w \in \mathcal{W}(a_1,a_2,a_3)} q_1^{s_1(w)} q_2^{s_2(w)} q_3^{s_3(w)} \quad .$$

Note that $F(a_1, a_2, a_3)(1, 1, 1) = |\mathcal{W}(a_1, a_2, a_3)| = (a_1 + a_2 + a_3)!/(a_1!a_2!a_3!)$.

Given a Laurent polynomial in the variables q_1, q_2, q_3, let's call it $P(q_1, q_2, q_3)$, let $POS(P)$ be the sum of the monomials all whose powers are strictly positive, i.e.

$$POS(\sum_{(i_1,i_2,i_3)} a_{i_1,i_2,i_3} q_1^{i_1} q_2^{i_2} q_3^{i_3}) := \sum_{(i_1,i_2,i_3), i_1>0, i_2>0, i_3>0} a_{i_1,i_2,i_3} q_1^{i_1} q_2^{i_2} q_3^{i_3} \quad .$$

For example

$$POS(5q_1^{-1} q_2^3 q_3^5 + 4q_1^2 q_2^{-3} q_3^5 + 7q_1 q_2 q_3^2 + 11q_1^2 q_3^3 + 2q_1 q_2 q_3) = 7q_1 q_2 q_3^2 + 2q_1 q_2 q_3 \quad .$$

Defining

$$G(a_1, a_2, a_3)(q_1, q_2, q_3) = POS(F(a_1, a_2, a_3)(q_1, q_2, q_3)) \quad ,$$

the desired number of sucker's bets with three decks of sizes a_1, a_2, a_3 and with cards carrying denominations $\{1, 2, \dots, a_1 + a_2 + a_3\}$, is

$$G(a_1, a_2, a_3)(1, 1, 1) \quad .$$

How to compute $F(a_1, a_2, a_3)$ (and hence $G(a_1, a_2, a_3)$, and hence $G(a_1, a_2, a_3)(1, 1, 1)$)?

Every non-empty word w either ends with a '1', or with a '2', or with a '3', hence the set of words in $1^{a_1} 2^{a_2} 3^{a_3}$, $\mathcal{W}(a_1, a_2, a_3)$, may be written as

$$\mathcal{W}(a_1, a_2, a_3) = \mathcal{W}(a_1 - 1, a_2, a_3)\, 1 \cup \mathcal{W}(a_1, a_2 - 1, a_3)\, 2 \cup \mathcal{W}(a_1, a_2, a_3 - 1)\, 3 \quad .$$

Of course $\mathcal{W}(a_1, a_2, a_3)$ is the empty set if any of a_1, a_2, a_3 is negative, and $\mathcal{W}(0, 0, 0)$ consists of one element, the **empty word**.

For any word $w \in \mathcal{W}(a_1 - 1, a_2, a_3)$ (given a word w and a letter l, wl is the word obtained from w by appending l to it)

$$s_1(w1) = s_1(w) + a_2 \quad , \quad s_2(w1) = s_2(w) \quad , \quad s_3(w1) = s_3(w) - a_3 \quad .$$

Similarly, for any word $w \in \mathcal{W}(a_1, a_2 - 1, a_3)$,

$$s_1(w2) = s_1(w) - a_1 \quad , \quad s_2(w2) = s_2(w) + a_3 \quad , \quad s_3(w2) = s_3(w) \quad ;$$

and, for any word $w \in \mathcal{W}(a_1, a_2, a_3 - 1)$,

$$s_1(w3) = s_1(w) \quad , \quad s_2(w3) = s_2(w) - a_2 \quad , \quad s_3(w3) = s_3(w) + a_1 \quad .$$

It follows that,

- For $w \in \mathcal{W}(a_1 - 1, a_2, a_3)$ we have

$$weight(w1) = weight(w) \cdot q_1^{a_2} q_3^{-a_3} \quad ;$$

- For $w \in \mathcal{W}(a_1, a_2 - 1, a_3)$ we have

$$weight(w2) = weight(w) \cdot q_1^{-a_1} q_2^{a_3} \quad ;$$

- For $w \in \mathcal{W}(a_1, a_2, a_3 - 1)$ we have

$$weight(w3) = weight(w) \cdot q_2^{-a_2} q_3^{a_1} \quad .$$

It follows that the Laurent polynomials $F(a_1, a_2, a_3) = F(a_1, a_2, a_3)(q_1, q_2, q_3)$ satisfy the **recurrence relation**

$$F(a_1, a_2, a_3) \; = \; q_1^{a_2} q_3^{-a_3} F(a_1 - 1, a_2, a_3) + q_1^{-a_1} q_2^{a_3} F(a_1, a_2 - 1, a_3) + q_2^{-a_2} q_3^{a_1} F(a_1, a_2, a_3 - 1) \quad , \qquad (Qrecurrence)$$

subject to the *boundary conditions* $F(0, 0, 0) = 1$ and $F(a_1, a_2, a_3) = 0$ if $a_1 < 0$ or $a_2 < 0$ or $a_3 < 0$.

This recurrence was programmed in Maple, and from this we deduced the positive parts, $G(a_1, a_2, a_3)$, and plugging-in $q_1 = 1, q_2 = 1, q_3 = 1$ we obtained

Important Fact: The first 12 terms of the sequence

'number of sucker's bets' with a set of three decks, each with n cards, where the $3n$ cards have all different numbers (labeled $1, \dots, 3n$), starting at $n = 1$, are

$$0, 0, 15, 39, 5196, 32115, 2093199, 19618353, 960165789, 11272949151, 479538890271, 6504453085104 \quad .$$

Dividing by 3 to account for trivial cyclic symmetry, the reduced numbers are

$$0, 0, 5, 13, 1732, 10705, 697733, 6539451, 320055263, 3757649717, 159846296757, 2168151028368 \quad .$$

It follows, that the sequence of probabilities for a random set of 3 decks of cards each with n cards to be a sucker's bet set, for n from 1 to 12 are

$$0., 0., 0.008928571429, 0.001125541126, 0.006866149723, 0.001872252397, 0.005245153668,$$

$$0.002072614083, 0.004213592531, 0.002030797274, 0.003512410777, 0.001921704153 \quad .$$

From Counting to Listing using Symbol Crunching

Suppose that we actually want to see *all* the possible sets of three decks that are sucker's bets? It is probably beyond the scope of computer-kind to list all 6504453085104 sucker's bets with 3 decks of 12 cards each, but a minor tweak to the recurrence ($Qrecurrence$) will enable us to do it for smaller sizes (and in principle, for all sizes).

First we need to tweak the definition of *weight*

$$weightX(w) := q_1^{s_1(w)} q_2^{s_2(w)} q_3^{s_3(w)} \cdot \prod_{i=1}^{|w|} x[i, w_i] \quad . \qquad (WeightX)$$

This keeps track of the individuality of the word. Next we define a Laurent polynomial in q_1, q_2, q_3 **and** polynomial in the indeterminates $x[i, j]$ ($1 \leq i \leq |w|$, $j \in \{1, 2, 3\}$)

$$F_X(a_1, a_2, a_3)(q_1, q_2, q_3, \{x[i, j]\}) := \sum_{w \in \mathcal{W}(a_1, a_2, a_3))} weightX(w) \quad .$$

The same argument that lead to ($Qrecurrence$) leads to

$$F_X\,(a_1, a_2, a_3) \, =$$

$$x[a_1 + a_2 + a_3, 1]\, q_1^{a_2} q_3^{-a_3} F_X\,(a_1 - 1, a_2, a_3)$$

$$+\, x[a_1 + a_2 + a_3, 2]\, q_1^{-a_1} q_2^{a_3} F_X\,(a_1, a_2 - 1, a_3)$$

$$+\, x[a_1 + a_2 + a_3, 3]\, q_2^{-a_2} q_3^{a_1} F_X\,(a_1, a_2, a_3 - 1) \quad , \qquad (Xrecurrence)$$

subject to the *boundary conditions* $F(0, 0, 0) = 1$ and $F(a_1, a_2, a_3) = 0$ if $a_1 < 0$ or $a_2 < 0$ or $a_3 < 0$.

We are only interested in the part where all the powers of the variables q_1, q_2, q_3 are positive, so let's apply POS:

$$G_X(a_1, a_2, a_3)(q_1, q_2, q_3, \{x[i, j]\}) \, = \, POS(F_X(a_1, a_2, a_3)(q_1, q_2, q_3, \{x[i, j]\})) \quad .$$

Now each monomial corresponds to a 'sucker's bets word'. The computer automatically transcribes each such monomial to a set of three decks of cards.

See the output file

- `http://sites.math.rutgers.edu/~zeilberg/tokhniot/oSuckerBets3.txt` ,

for the five such sets (up to trivial cyclic symmetry) with 3 decks, each with 3 cards.

- `http://sites.math.rutgers.edu/~zeilberg/tokhniot/oSuckerBets4.txt` ,

for the thirteen such sets (up to trivial cyclic symmetry) with 3 decks, each with 4 cards.

- `http://sites.math.rutgers.edu/~zeilberg/tokhniot/oSuckerBets5.txt` ,

for the 1732 such sets (up to trivial cyclic symmetry) with 3 decks, each with 5 cards.

From Words to Lattice Paths

From words, in turn, we can get lattice walks, with unit positive steps, from the origin to the point $(a_1, \ldots, a_k)$, where $w_i = j$ corresponds to the i-th step being parallel to the x_j-axis. For example, the above ('magic-square' word), 132321213 . corresponds to the 3D lattice walk

$$(0,0,0) \rightarrow (1,0,0) \rightarrow (1,0,1) \rightarrow (1,1,1) \rightarrow (1,1,2) \rightarrow (1,2,2) \rightarrow (2,2,2) \rightarrow (2,3,2) \rightarrow (3,3,2) \rightarrow (3,3,3) \quad .$$

From now on the number of decks, k, is no longer 3.

The advantage of the lattice path representation is that it allows us to construct sucker's bets where repeated denominations are allowed. If one wants to avoid the possibility of a tie (like in *paper-scissors-stone*) then the allowed steps are always parallel to one of the k axes, but not necessarily unit steps. If one does not mind ties, then we can also have 'diagonal' steps, where more than one coordinate changes.

For example, the Efron set of four dice mentioned at the very beginning of this treatise corresponds to the following lattice path in the 4D hyper-cubic lattice, with seven steps

$$(0,0,0,0) \rightarrow (2,0,0,0) \rightarrow (2,0,0,3) \rightarrow (2,0,4,3) \rightarrow (2,6,4,3) \rightarrow (6,6,4,3) \rightarrow (6,6,4,6) \rightarrow (6,6,6,6) \quad .$$

One can easily define analogs of $s_i(w)$ for these more general walks (bets), and establish the generalizations of ($Qrecurrence$) and ($Xrecurrence$). See the Maple package `SuckerBets.txt` available directly from

`http://sites.math.rutgers.edu/~zeilberg/tokhniot/SuckerBets.txt` ,

or via the front of this treatise

`http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/suckerbets.html` ,

where there are links to plenty of input and output files.

The Efron set of dice, mentioned above, is a bit wasteful, and there exists a tie-less four-dice sucker's bet set only using **six** different denominations, see

`http://sites.math.rutgers.edu/~zeilberg/tokhniot/oSuckerBets0a.txt` .

Here it is:

$$A = [1,1,5,5,5,5] \quad , \quad B = [4,4,4,4,4,4] \quad , \quad C = [3,3,3,3,3,3] \quad , \quad D = [2,2,2,2,6,6] \quad .$$

This is the **unique** such set, up to trivial cyclic symmetry.

Efron's set is one of 38 different (tie-less, and up to trivial cyclic symmetry) sets with **seven** different denominations, viewable from

`http://sites.math.rutgers.edu/~zeilberg/tokhniot/oSuckerBets0b.txt` .

There are 755 different (tie-less, and up to trivial cyclic symmetry) sets with *eight* different denominations, viewable from

`http://sites.math.rutgers.edu/~zeilberg/tokhniot/oSuckerBets0c.txt` .

For many more examples, where the number of cards (or faces) in each deck (or die) are not necessarily the same, see the other outputs files.

Statistical Analysis

Now we are back to $k = 3$. What we do here for three-deck sets is generalizable to general $k \geq 3$, but things are already interesting (and complex enough) for the case $k = 3$.

The three individual *word-statistics* $s_1(w), s_2(w), s_3(w)$ are closely related, and in fact, trivially equivalent to, the so-called 'Number of inversions' introduced in the 19th century by Eugen Netto (of the *Lehrbuch* fame), and rediscovered, at the mid 20th century, by non-parametric statisticians H. B. Mann and D. R. Whitney ([MW]). It is well known (see, e.g. [CJZ]), that each of $s_1(w), s_2(w), s_3(w)$ is, *individually*, **asymptotically normal** , but of course, they are far from being independent.

We discovered that the *limiting scaled* tri-variate distribution of the triple (discrete) random variable $(s_1(w), s_2(w), s_3(w))$ defined on the set of words on $1^n 2^n 3^n$, namely $\mathcal{W}(n,n,n)$, as $n \to \infty$, converges, **in distribution**, to the **limit** as $c \to 1^-$ of the trivariate continuous random variable whose **joint density** function is

$$f(x,y,z;c) \ := \ \frac{exp(-x^2/2 \ - \ y^2/2 - z^2/2 \ - \ c\,(xy + xz + yz))}{N(c)} \quad ,$$

where $N(c)$ is the normalization factor that would make $\int_{R^3} f(x,y,z;c) = 1$, namely

$$N(c) := \frac{(2\pi)^{3/2}}{(1-c)\sqrt{1+2c}} \quad .$$

Note that this 'blows up' at $c = 1$.

Officially this is still a conjecture, but we are **absolutely sure** that it is correct, since we proved, **rigorously**, that all the scaled mixed moments, $M(i_1, i_2, i_2)$, of the triple of random variables (s_1, s_2, s_3), converge as $n \to \infty$ to the limit of the corresponding scaled mixed moments of $f(x, y, z; c)$, as $c \to 1^-$, for all $1 \le i_1, i_2, i_3 \le 5$, and with more computing power, one can easily go further.

We are offering to donate 100 dollars to the OEIS, in honor of the first prover of our conjecture. (Not because we have any doubts about its truth, or that we particularly care about so-called rigorous proofs, but because we love the OEIS!)

Explicit (Rigorously Proved!) Expressions for Mixed Moments of (s_1, s_2, s_3)

See: `http://sites.math.rutgers.edu/~zeilberg/tokhniot/oSuckerBetsAnalysis4LP.txt` .

Let's just summarize a few highlights.

The variance of each of s_1, s_2, s_3 on $\mathcal{W}(n, n, n)$ is $n^2\,(2\,n + 1)\,/3$, and the kurtosis is

$$\frac{3(10n^2 - n - 4)}{5n(2n + 1)} \quad .$$

Note that it tends to 3, as n goes to infinity, as it should, since it is *asymptotically normal.* This is all old stuff (see [CJZ]), as well as any of the moments of each of the single random variables s_1, s_2, s_3.

The *covariance* between any pair of s_1, s_2, s_3 is very simple, it is

$$-\frac{n^3}{3} \quad ,$$

hence the *correlation is*

$$-\frac{n}{2n + 1} \quad ,$$

that converges to $-\frac{1}{2}$.

Explicit expressions for all the mixed moments can be found in the above output file.

Let's just write down, in humanese, the $(4, 5, 5)$ mixed moment, that happens to be a polynomial in n of degree 21.

$$\frac{1}{2837835}\,n^3(39239200\,n^{18} + 66146080\,n^{17} - 816055240\,n^{16}$$

$$+1114633520\,n^{15} + 3208398492\,n^{14} - 13589761044\,n^{13} + 25028291837\,n^{12} - 38043392560\,n^{11} + 62580129596\,n^{10}$$

$$-103184180072\,n^9 + 157753326632\,n^8 - 224678523360\,n^7 + 293133737664\,n^6 - 336053442624\,n^5$$

$$+322828696448\,n^4 - 243844376832\,n^3 + 132045454336\,n^2 - 44452356096\,n + 6864979968) \quad .$$

Here are all the scaled limits, $S(i_1, i_2, i_3)$, for $0 \leq i_1, i_2, i_3 \leq 5$. By symmetry we only need to list the values for $i_1 \leq i_2 \leq i_3$. Note that $S(i_1, i_2, i_3) = 0$ when $i_1 + i_2 + i_3$ is odd.

$$S(0,0,0) = 1 \quad , \quad S(0,0,2) = 1 \quad , \quad S(0,0,4) = 3 \quad , \quad S(0,1,1) = -1/2 \quad ,$$

$$S(0,1,3) = -3/2 \quad , \quad S(0,1,5) = -15/2 \quad , \quad S(0,2,2) = 3/2 \quad , \quad S(0,2,4) = 6 \quad ,$$

$$S(0,3,3) = -21/4 \quad , \quad S(0,3,5) = -30 \quad , \quad S(0,4,4) = 57/2 \quad , \quad S(0,5,5) = -765/4 \quad ,$$

$$S(1,1,2) = 0 \quad , \quad S(1,1,4) = 3/2 \quad , \quad S(1,2,3) = -3/4 \quad , \quad S(1,2,5) = -15/2 \quad ,$$

$$S(1,3,4) = 3/2 \quad , \quad S(1,4,5) = -45/4 \quad , \quad S(2,2,2) = 3/2 \quad , \quad S(2,2,4) = 6 \quad ,$$

$$S(2,3,3) = -3 \quad , \quad S(2,3,5) = -45/2 \quad , \quad S(2,4,4) = 45/2 \quad , \quad S(2,5,5) = -135 \quad ,$$

$$S(3,3,4) = 0 \quad , \quad S(3,4,5) = -135/4 \quad , \quad S(4,4,4) = 135/2 \quad , \quad S(4,5,5) = -945/4 \quad .$$

They do indeed coincide with the llimits as $c \to 1^-$ of the scaled moments of $f(x,y,z;c)$. The mixed moments of the latter can be computed as far as desired thanks to linear recurrences of order 4 found via the Apagodu-Zeilberger multi-variable Almkvist-Zeilberger algorithm ([AZ]). When one plugs-in $c = 1$ you still get complicated recurrences, but the **diagonal**, the mixed moments $(2k, 2k, 2k)$ are given by a very nice closed form

$$S(2k, 2k, 2k) \;=\; \frac{(3\,k)!\;(2\,k)!}{8^k\,(k!)^2} \quad .$$

How To Find Rigorously Proved Polynomial Expressions for the Mixed Moments of (s_1, s_2, s_3)?

We use the approach of [BZ] and [Z]. Suppose that we are interested in an explicit expression for a certain specific mixed moment $M(i,j,k)$ of the triple (s_1, s_2, s_3) defined on $\mathcal{W}(n,n,n)$. We know, *a priori*, that it is a certain *polynomial* in n, and we can easily bounds its degree, let's call it d. Hence it suffices to find $d+1$ terms. But $M(i,j,k)$ at (n,n,n) is

$$\frac{n!^3}{(3n)!} \cdot (q_1 \frac{\partial}{\partial q_1})^i (q_2 \frac{\partial}{\partial q_2})^j (q_3 \frac{\partial}{\partial q_3})^k \, F(n,n,n)(q_1,q_2,q_3)\Big|_{q_1=1, q_2=1, q_3=1} \quad .$$

We use $(Qrecurrence)$ to compute enough terms $F(n,n,n)$ for $n \leq d+1$, and then 'fit the data'.

The drawback of the above approach is that it is inefficient. If we are only interested in the first few mixed moments, then we can do a tri-variate Taylor expansion around $(q_1, q_2, q_3) = (1,1,1)$ and truncate anything beyond our horizon. This is accomplished by using $(Qrecurrence)$, with $q_1 = 1 + p_1, q_2 = 1 + p_2, q_3 = 1 + p_3$ and doing the truncated version. This gives a quicker way to find the *mixed factorial moments* from which the mixed moments can be easily gotten. Full details can be gotten by reading the Maple source code available from the front of this article mentioned above.

Conclusion

As with most of our research, the *methodology* is at least as interesting as the actual results. The present project is a **case study** of using *symbol crunching* and *experimental mathematics* both to generate interesting combinatorial objects (in this case *sucker's bets*), via *symbolic dynamical programming*, and to do *symbolic statistical analysis* about intriguing combinatorial random variables, that are very negatively correlated, and for which there is a beautiful *asymptotic* limiting trivariate distribution, namely the limit, as c goes to 1 from the below, of the one whose joint density function is

$$f(x,y,z;c) \,:=\, \frac{exp(-x^2/2\,-\,y^2/2\,-\,z^2/2\,-\,c\,(xy+xz+yz))}{\frac{(2\pi)^{3/2}}{(1-c)\sqrt{1+2c}}} \quad .$$

References

[AZ] Moa Apagodu and Doron Zeilberger, *Multi-Variable Zeilberger and Almkvist-Zeilberger Algorithms and the Sharpening of Wilf-Zeilberger Theory*, Adv. Appl. Math. **37**(2006), (Special issue in honor of Amitai Regev), 139-152.
`http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/multiZ.html`

[BZ] Andrew Baxter and Doron Zeilberger, *The Number of Inversions and the Major Index of Permutations are Asymptotically Joint-Independently Normal (2nd ed.)*, Personal Journal of Shalosh B. Ekhad and Doron Zeilberger, Feb. 4, 2011.
`http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/invmaj.html` .

[CJZ] E. Rodney Canfield, Svante Janson, and Doron Zeilberger, *The Mahonian Probability Distribution on Words is Asymptotically Normal*, Advances in Applied Mathematics **46**(2011), 109-124. Available from:
`http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/mahon.html` ,
erratum:
`http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/mahonerratum.pdf` .

[G] Martin Gardner, *"The Colossal Book of Mathematics: Classic Puzzles, Paradoxes, and Problems: Number Theory, Algebra, Geometry, Probability, Topology, Game Theory, Infinity, and Other Topics of Recreational Mathematics"* (1st ed.), New York: W. W. Norton & Company..

[MW] H. B. Mann and D. R. Whitney, *On a test whether one of two random variables is stochastically larger than the other.* Annals of Mathematical Statistics **18**(1947), 50-60.

[NZ] John Noonan and Doron Zeilberger, *"The Goulden-Jackson Cluster Method: Extensions, Applications, and Implementations"*, J. Difference Eq. Appl. **5** (1999), 355-377.
`http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/gj.html` .

[P] W. Penney, *Problem 95: Penney ante*, J. Recreational Mathematics **7**(1974), 321.

[W] The Wikipedia Foundation, *"Nontransitive Dice"*,
`https://en.wikipedia.org/wiki/Nontransitive_dice` .

[Z] Doron Zeilberger, *Doron Gepner's Statistics on Words in* $\{1,2,3\}^*$ *is (Most Probably) Asymptotically Logistic*. Personal Journal of Shalosh B. Ekhad and Doron Zeilberger, March 31, 2016.
`http://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/gepner.html` .

Doron Zeilberger, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.
DoronZeil at gmail dot com ; `http://sites.math.rutgers.edu/~zeilberg/` .

Shalosh B. Ekhad, c/o D. Zeilberger, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.

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