Computational and Theoretical Challenges On Counting Solid Standard Young Tableaux

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Very Important: This article is accompanied by the Maple package `http://www.math.rutgers.edu/~zeilberg/tokhniot/SolidSYT` . Readers who have access to Maple should download it and read it into a Maple session, so that they can follow the text more vividly.

The subject of *solid partitions* goes back to Percy MacMahon [M], but to our surprise, as far as we can tell by *googling*, no one has seriously studied *Solid Standard Young Tableaux*. Let's first recall some basic facts about the familiar kind.

Review of (2D) Standard Young Tableaux

Recall that a (usual) (2D) *Young diagram* of shape $\lambda = (\lambda_1, \dots, \lambda_k)$ (where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$ are integers) is a left-justified collection of k rows of empty unit-boxes, where the top row has λ_1 boxes, the second row has λ_2 boxes, ..., and the bottom, k-th row, has λ_k boxes. For example, the following is a Young diagram of shape $(3, 2, 2, 1)$

$$\begin{array}{lll} X & X & X \\ X & X & \\ X & X & \\ X & & \end{array} \quad ,$$

where X denotes an empty unit-box. Let $n := \lambda_1 + \dots + \lambda_k$ be the number of boxes (alias the "integer that is being partitioned" by λ). A *Standard Young Tableau* is a way of placing the integers 1 through n inside the boxes, so that **all rows** and **all columns** are **increasing** (when read from left-to-right, and top-to-bottom respectively). For example,

$$\begin{array}{lll} 1 & 3 & 8 \\ 2 & 5 & \\ 4 & 6 & \\ 7 & & \end{array} \quad ,$$

is one of the seventy Standard Young Tableaux of Shape $(3, 2, 2, 1)$. There is a beautiful formula, due to Frame, Robinson, and Thrall[FRT], called the *hook-length formula*, for the number of Standard Young Tableaux of a given shape. Calling that number f_λ, it is:

$$f_\lambda = \frac{n!}{\prod_{b \in \lambda} h_b} \quad ,$$

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where for any *box* b in the Young diagram, h_b is its *hook-length*, i.e. the number of boxes that are either weakly to its right or weakly below. For example, the hook-lengths for the shape $(3,2,2,1)$ are

$$\begin{matrix} 6 & 4 & 1 \\ 4 & 2 & \\ 3 & 1 & \\ 1 & & \end{matrix} \quad .$$

It follows, that

$$f_{3221} = \frac{8!}{6 \cdot 4 \cdot 1 \cdot 4 \cdot 2 \cdot 3 \cdot 1 \cdot 1} = 70 \quad ,$$

as claimed above.

The **proof from the book** of that amazing formula is due to Curtis Greene, Albert Nijenhuis, and Herbert Wilf[GNW]. In addition to its considerable *face-value*, this proof is also historically significant, since it turned DZ from an unhappy analyst into a happy combinatorialist!

The [GNW] *proof-from-the-book* is presumbly a serendipitous *by-product* of the Greene-Nijenhuis-Wilf *algorithm-from-the-book* (also told in [GNW]) to generate a Standard Young Tableau of a given shape *uniformly at random*. It goes like this.

First roll an n-faced *fair* die, and decide accordingly the starting box. Then whenever visiting a box b, roll a fair $(h_b - 1)$-faced die, and decide which box in the hook (except the one you are at right now, you must move on!) to go to next. Keep doing it, until you wind-up at a corner, where there is nowhere to go. Put n there. Now you have a smaller shape, with $n-1$ empty boxes, and continue recursively.

[This algorithm is implemented in procedure `GNW` of the Maple package
`http://www.math.rutgers.edu/~zeilberg/tokhniot/GreeneNijenhuisWilf`] .

The total number of Standard Young Tableaux with n cells, $\omega_n := \sum_{\lambda \dashv n} f_\lambda$ is the famous "number of involutions", sequence `http://oeis.org/A000085` , (thanks to the so-called *Robinson-Schenstead algorithm*), that has a very simple, *recurrence* $\omega_n = \omega_{n-1} + (n-1)\omega_{n-2}$, that enables one to easily compute the first ten thousand terms in a fraction of a second.

Solid Standard Young Tableaux

Now the shapes are the 3D Young diagrams of *plane partitions*. Recall that a *plane partition* is a two-dimensional array of positive integers p_{ij} where both rows and columns are *weakly-decreasing* and its Young diagram consists of piling p_{ij} empty boxes above location (i,j) on the floor. A *Solid Standard Young Tableau* of a given shape (with n empty boxes) is a way of placing the integers 1 through n such that going from left-to-right, from back-to-front, and from down-to-up the entries are **increasing**.

It is unrealistic to expect a *nice* formula for the number of Solid Standard Young Tableaux of a given 3D shape, but we can still easily compute it, using an obvious recurrence. If the number of boxes in our shape is n, then the entry n must reside in one of the *corners* (unit-boxes where all the forward-going neighbors (in each of the three directions) do not belong to the shape). Then f_λ (where λ is now a 3D shape (alias a plane-partition)) is the sum of $f_{\lambda'}$ over all λ' obtained by removing a corner box from λ. This is implemented in `SolidSYT`'s procedure `Nu(L)`, where L is a shape (i.e. plane partition), expressed as a list-of-lists of positive integers.

The 3D analog of ω_n, obtained by summing $Nu(L)$ over all plane partitions L of a given integer n, is implemented by procedure `Y3number(n);`, and to get the first K terms of that sequence, type `Y3numberSeq(K);`.

For the record, here are the first thirty terms, [taken from `http://www.math.rutgers.edu/~zeilberg/tokhniot/oSolidSYT1`]:
$1, 3, 9, 33, 135, 633, 3207, 17589, 102627, 636033, 4161141, 28680717, 207318273, 1567344549, 12345147705, 101013795753, 856212871761, 7501911705747, 67815650852235, 631574151445665, 6051983918989833, 59605200185016639, 602764245172225251, 6252962956009863363, 66482211459036254169, 723810526382641418667, 8062440364611311185977, 91804267420894431624357, 1067720130017504052805449, 12673922788286515247094267$

Generating a Uniformly-at-Random Solid Standard Young Tableau of a Given Shape

Using the beautiful approach of [W], clearly explained and exploited in [NW], one can generate *uniformly at random*, a Solid Standard Young Tableau of a given (solid) shape. If λ is such a shape (alias plane partition) with n boxes, then the entry n can reside in any of its *corners* (boxes where none of its forward-going neighbors are in the shape). Let the set of corners be C. Then we form a **loaded die** whose faces are labeled by the members of C, and the probability of it lending on face c is $f_{\lambda-c}/f_\lambda$, where f_λ is the number of Solid Standard Young Tableaux of shape λ (implemented by procedure `Nu(L)` in `SolidSYT`). We then place the n in the corner-box decided by the die, and get a smaller shape, $\lambda' = \lambda - c$ with $n-1$ boxes, and continue recursively, until we get the empty shape.

[Procedure `RSSYT(L);` implements this in the Maple package `SolidSYT`, try for example, `RSSYT([[3,3,3],[3,3,3],[3,3,3]]);` for getting, *uniformly-at-random*, one of the 6405442434150 ways of placing 1 through 27 in a $3 \times 3 \times 3$ box, in such a way that when you go form left-to-right, from back-to-front, and from down-to-up, they are always increasing.]

The Three-Dimensional Greene-Nijenhuis-Wilf Algorithm

As the shapes get larger, `RSSYT` gets slower and slower, since it relies on the recursive procedure `Nu` (there is no (known) closed-form expression for f_λ for three-dimensional shapes).

The beauty of the Greene-Nijenhuis-Wilf algorithm is that it is so much faster! The die cast at every step is always fair! Unfortunately, the three-dimensional analog no longer gives you a random Solid Standard Young Tableau *uniformly*, but is gets (experimentally) fairly close. So if you don't mind a little bias, you are welcome to use procedure `GNW3(L);` .

For example

`GNW3([[10$10]$10]);` gives you, instantaneously, a (not-quite-uniformly-at) random way of placing 1 through 1000 in a $10 \times 10 \times 10$ box, in such a way that when you go form left-to-right, from back-to-front, and from down-to-up, they are always increasing.

Some Computational Challenges Regarding the Enumeration of Solid Standard Young-Tableaux of Cylindrical Shapes

Most of us know that the number of (usual, 2D) Standard Young Tableaux of shape (n, n) is given by the famous *Catalan Numbers* $(2n)!/(n!(n+1)!)$, `http://oeis.org/A000108` . The number of Standard Young Tableaux of shape (n, n, n) is given by the so-called *three-dimensional Catalan Numbers*, `http://oeis.org/A005789` , that count the number of ways of walking n steps in a 3D Manhattan always staying in $x \geq y \geq z$, and (as of Feb. 18, 2012) Sloane has it up to the five-dimensional version `http://oeis.org/A005791` .

Since (what Sloane calls) the k-dimensional Catalan numbers are given by the explicit (hypergeometric!) formula $(k-1)!(nk)!/(n! \cdots (n+k-1)!)$ (that follows immediately from the hook length formula) it follows that the enumerating sequence $a(n)$ (for each specific k) is a *hypergeometric* sequence, in other words, there exist polynomials $p_1(n)$ and $p_0(n)$ such that

$$p_0(n)a(n) + p_1(n)a(n+1) = 0 \quad ,$$

which is a special case (first-order) of a very important *ansatz*, the so-called *holonomic*, or *P-recursive* ansatz, that consists of sequences satisfying a linear-recurrence equation with polynomial coefficients of some (finite) order L:

$$\sum_{i=0}^{L} p_i(n)a(n+i) = 0 \quad .$$

Going to sequences enumerating Solid Standard Young Tableaux of Cylindrical Shapes, i.e. $\lambda \times \{1, \ldots, n\}$, for a (usual) partition λ, things are very mysterious. For $\lambda = (2, 1)$, i.e. for 3D-shapes of the form $[[n, n], [n]]$ we have (not-quite-so-trivially!) the famous *Kreweras sequence*, `http://oeis.org/A006335` (why?), that is also a hypergeometric sequence, and hence holonomic.

Procedure `Sidra(L,n,N0);` in our Maple package `SolidSYT` spits out the first `N0` terms of the enumerating sequence for a family of shapes L with parameter n. For example, to get the first 20 terms of the Kreweras sequence type: `Sidra([[n,n],[n]],n,20);` .

On the other hand, if $\lambda = [[n, n], [n, 1]]$, the sequence "`Sidra([[n,n],[n,1]],n,40);` " can be "described" (empirically, so far) by a second-order linear recurrence equation with polynomial coefficients, see:

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oSolidSYT3` .

1st Rigorous Challenge (0.01 US dollars) : Find a rigorous proof of this recurrence .

To our surprise, the enumerating sequence for the number of Solid Standard Young Tableaux of the cylindrical shapes $(2,1,1) \times \{1, \dots n\}$, i.e., in the notation of `SolidSYT`,

```
Sidra([[n,n],[n],[n]], n, BigEnough);
```

for which, with some effort, (with `BigEnough`=120), we were able to find the first 121 terms, see

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oSolidSYT7` ,

did not yield a linear recurrence equation with polynomial coefficients of order `ORDER` and degree `DEGREE` with `(ORDER+1)(DEGREE+1)` less than 115. This brings us to the:

1st Non-Rigorous Challenge (100 US dollars) : Find a linear recurrence equation with polynomial coefficients (empirically) satisfied by the sequence $a(n) :=$ "number of Solid Standard Young Tableaux" of shape $(2,1,1) \times \{1, \dots n\}$ (or equivalently, $(3,1) \times \{1, \dots n\}$). Equivalently, $a(n)$ is the number of ways of walking from $(0,0,0,0)$ to (n,n,n,n) using positive unit steps in the four-dimensional Manhattan lattice, in such a way that all the visited points (x_1, x_2, x_3, x_4) always satisfy $x_1 \geq x_2 \geq x_3$ **and** $x_1 \geq x_4$.

2nd Rigorous Challenge (1 US dollar) : Having "conjectured" the above recurrence (i.e. proven it experimentally), find a "rigorous" proof.

Feb. 28, 2012 Update: Manuel Kauers and his student, Fredrik Johansson, kindly informed us that no recurrence exists with `(ORDER+1)(DEGREE+1)` less than 3000. See Manuel Kauers's message in:
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/ssytAppendix.html` .

If the answer is negative, then we have the alternative

2'nd Rigorous Challenge (10 US dollars) : Prove that the above sequence is **not** holonomic. Finally, the

Big Question: (Lots of glory but no cash) Characterize all partitions λ for which the sequence enumerating Solid Standard Young Tableaux of shape $\lambda \times \{1, \dots, n\}$, $(n = 1, 2, \dots)$ satisfy a linear recurrence equation with polynomial coefficients.

References

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