The Generating Functions Enumerating 12...d-Avoiding Words with r occurrences of each of 1 , . . . , n are D-finite for all d and all r

By Shalosh B. EKHAD and Doron ZEILBERGER

Dedicated to Neil James Alexander Sloane (born October 10, 1939), on his A000027[75]-th birthday (alias A005408[38]-th, alias A002808[53]-th, alias A001477[74]-th, alias A014612[17]-th, and 13520 other aliases.)

Introduction

In a recent beautiful article, Nathaniel Shar and Doron Zeilberger ([ShZ]) proved that for any positive integer r, the generating function of the sequence enumerating 123-avoiding words with r occurrences of each of the letters $1, \dots, n$ is always **algebraic**. In other words for each r, the generating function, let's call it $f_r(x)$, satisfies an equation of the form

$$P_r(x, f_r(x)) = 0 \quad ,$$

for *some* polynomial, P_r, of two variables. The actual polynomials, $P_r(x, y)$, were computed for $r \leq 4$.

This is no longer true for $12 \dots d$-avoiding words with $d \geq 4$, *even* for $r = 1$.

In 1990 Doron Zeilberger ([Z]) showed that for and *any* positive integer d, the generating function enumerating $1 \dots d$-avoiding *permutations* (i.e. words in $\{1, \dots, n\}$ where each letter occurs exactly 1 times) is the next-best-thing to being algebraic, which is being **D-finite** (aka as *holonomic*). Recall that a formal power series is D-finite if it satisfies a linear *differential* equation with polynomial coefficients, or equivalently, the enumerating sequence itself is **P-recursive**, i.e. satisfies a linear *recurrence* equation with polynomial coefficients. Ira Gessel ([G]) famously discovered (and proved) a beautiful determinant with Bessel functions, for the generating function, (of the sequence divided by $n!^2$) (that also implies the above result), and Amitai Regev ([R]) famously derived delicate and precise asymptotics.

In the present article, dedicated to guru Neil Sloane on his 75-th birthday, we observe that the analogous generating functions for *multi-set permutations* (alias *words*), where every letter appears the same number of times, say r, are still always **D-finite**, (for *every* d and *every* r), and we actually crank out the first few terms of quite a few of them, many of whom are not yet in the OEIS ([Sl]).

All this data, often with linear recurrences (that we know *a priori* exist, and hence it justifies their discovery by pure guessing), and very precise asymptotics, is collected in the front of this article

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/sloane75.html` where links to two useful Maple packages, that were used to generate all that data, `SLOANE75` and `NEIL`, can be found and downloaded, and readers who have Maple and computer time to spare are welcome to use in order to generate yet more data.

Last but not least, we pledge 100 dollars to the OEIS in honor of the first one to prove our conjectured asymptotic formula for the number of $1 \dots d$-avoiding words in $\{1^r \dots n^r\}$ that generalizes Regev's ([R]) famous formula for $r = 1$. We pledge another 100 dollars for extending Gessel's Bessel determinant, from the $r = 1$ case to general r.

Why is the Sequence Enumerating $\mathbf{1 \dots d}$-avoiding words in $\{\mathbf{1^r \dots n^r}\}$ P-recursive?

By the Robinson-Schenstead-Knuth (RSK) famous correspondence, our quantity of interest, let's call it $A_{d,r}(n)$ is given by

$$A_{d,r}(n) \, = \sum_{\substack{\lambda \vdash rn \\ length(\lambda) \leq d}} f_\lambda g_\lambda^{(r)} \quad ,$$

where f_λ is the number of standard Young tableaux of shape $\lambda = (\lambda_1, \dots, \lambda_d)$, and $g_\lambda^{(r)}$ is the number of column-strict Young tableaux with exactly r occurrences of each of $1, \dots, n$. For $\lambda = (\lambda_1, \dots, \lambda_d)$ (where we pad it with zeroes if the length is less than d), f_λ is closed-form (thanks to Young-Frobenius, or the hook-length formula), and hence *ipso facto*, holonomic in its d discrete arguments. Furthermore, for $r > 1$, $g_\lambda^{(r)}$, while no longer closed-form, is easily seen to be holonomic in its d discrete arguments (one way to see this is to note that their *redundant* generating function (in the sense of MacMahon) is a rational formal power series in $x_1, \dots, x_d$). It follows, by *general holonomic nonsense* ([Z]), that for any **fixed** integers r and d the sequence, in n, $\{A_{d,r}(n)\}$, is P-recursive. Computationally speaking, it is fairly easy to compute $g_\lambda^{(r)}$, and hence crank-out the first few terms of the sequences $\{A_{d,r}(n)\}$ for quite a few d and r, that for r and d not too large may be used to guess (in *real* time) the recurrences empirically, that we know must be the right ones.

The 100 dollars conjecture generalizing Regev's Asymptotics

Conjecture (100 donation to the OEIS in honor of the first prover)

Let $A_{d,r}(n)$ be the number of $1 \dots d$-avoiding words in $\{1^r \dots n^r\}$, then there exists a constant $C_{r,d}$ such that

$$A_{d,r}(n) \, \sim \, C_{r,d} \cdot \left(\binom{d+r-2}{d-2} (d-1)^r \right)^n \cdot \frac{1}{n^{((d-1)^2-1)/2}} \quad .$$

Extra Credit (25 additional dollars): find an explicit expression for $C_{r,d}$ in terms of r and d (involving π, of course).

The 100 dollars Challenge to generalize Gessel's Spectacular Theorem

This is more open-ended, but it would be nice to get a determinant expression, in the style of Ira Gessel's ([G]) famous expression for the generating function of $A_{d,1}(n)/n!^2$, *canonized* in the *bible*

([W], p. 996, Eq. (5)). Here it is: Let $u_k(n) := A_{k,1}(n)$, then

$$\sum_{n \geq 0} \frac{u_k(n)}{n!^2} \, x^{2n} = \det(I_{|i-j|}(2x))_{i,j=1,\dots,k} \quad ,$$

in which $I_\nu(t)$ is (the modified Bessel function)

$$I_\nu(t) = \sum_{j=0}^{\infty} \frac{(\frac{1}{2}\, t)^{2j+\nu}}{j!(j+\nu)!} \quad .$$

Guru Herb Wilf (ibid) goes on to wax eloquently:

" At any rate, it seems fairly "spectacular" to me that when you place various infinite series such as the above into a $k \times k$ determinant, and then expand the determinant, you should find that the coefficient of x^{2n}, when multiplied by $n!^2$, is exactly the number of permutations of n letters with no increasing subsequence longer than k ."

It would be even more spectacular, if you, dear reader, would generalize this to $r > 1$!

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EXCLUSIVELY PUBLISHED IN The Personal Journal of Shalosh B. Ekhad and Doron Zeilberger (http://www.math.rutgers.edu/~zeilberg/pj.html) and arxiv.org.

Dec. 5, 2014