Searching for Strange Hypergeometric Identities By Sheer Brute Force

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Important Note: This article is accompanied by the Maple package `BruteTwoFone` available from the webpage of this article
`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/sheer.html` ,
where one can also find sample input and output.

Preface

The classical *hypergeometric* series

$$F(a\,,\,b\,,\,c\,,\,x) \;=\; \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} x^k \quad ,$$

(where $(z)_k := z(z+1)(z+2)\cdots(z+k-1)$), that nowadays is more commonly denoted by

$${}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}\;;\;x\right) \quad ,$$

has a long and distinguished history, going back to Lehonard Euler and Carl Friedrich Gauss. It was also one of Ramanujan's favorites. Under the guise of **binomial coefficient sums** it goes even further back, to Chu, in his 1303 combinatorics treatise, that summarized a body of knowledge that probably goes yet further back.

The hypergeometric function, and its generalized counterparts, enjoy several *exact evaluations*, for some choices of the parameters, in terms of the Gamma function. The classical identities of Chu-Vandermonde, Gauss, Kummer, Euler, Pfaff-Saalschütz, Dixon, Dougall, and others, can be looked up in the *classic* classic text of Bailey[B], and the *modern* classic text of Andrews, Askey, and Roy [AAR].

For example, when $x = 1$, Gauss found the 3-parameter exact evaluation:

$$F(a\,,\,b\,,\,c\,,\,1) \;=\; \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad . \qquad (Gauss)$$

(When a is a negative integer, $a = -n$, then this goes back to Chu's 1303 identity, rediscovered by Vandermonde).

Next comes Kummer's *two-parameter* exact evaluation at $x = -1$

$$F(a\,,\,b\,,\,1+a-b\,,\,-1) = \frac{\Gamma(1+a-b)\Gamma(1+a/2)}{\Gamma(1+a)\Gamma(1+a/2-b)} \quad , \qquad (Kummer)$$

[1] Supported in part by the USA National Science Foundation.

and two other ones, at $x = \frac{1}{2}$, due to Gauss (see [B] or [AAR]). In addition, Gosper conjectured, and Gessel and Stanton[GS] proved, several 'strange' *one-parameter* evaluations at other values of x.

The Hypergeometric function also enjoys several *transformation formulas*, both rational and quadratic, due to Euler, and Pfaff (see [AAR], Theorem 2.2.5, Corollary 2.3.3 and Theorem 3.1.3), so any exact evaluation implies quite a few other ones, equivalent to it via these transformations, and iterations thereof. [See procedures `Buddies21C` and `QuadBuddies21`, in our Maple package `BruteTwoFone`.]

In [E], a systematic search for *all* such strange identities, up to a certain "complexity" was attempted. It was done by implementing the method of [Z], that used Wilf-Zeilberger theory and Zeilberger's algorithm as simplified in [MZ]. The drawback of that method, however, was that it only searched for those identities for which the Zeilberger algorithm outputs a *first-order* recurrence. While rare, there are cases where the Zeilberger algorithm outputs a higher-order recurrence, yet still is evaluable in closed-form. This is because this algorithm is **not** guaranteed to output the minimal-order recurrence, although it usually does.

This gave us the idea to systematically search for such closed-form evaluations by *sheer brute force*. Surprisingly, it lead us to the discovery of two *new* infinite families of "closed-form" evaluations, that we will describe later.

When a or b happen to be a negative integer, $a = -n$, say, then the *infinite* series **terminates**, for the sum then only has $n + 1$ terms, and one does not have to worry about convergence. In most cases, one can easily pass from the terminating case to the non-terminating case by Carlson's theorem ([AAR], p. 108; [B], p. 39). In this article we will only consider such terminating series.

The Haystack

In this *etude* in **Experimental Mathematics**, the haystack consists of

$$F(-an, bn + b_0, cn + c_0, x) \quad , \tag{2F1}$$

where a is a positive integer, b and c are integers, while b_0, c_0, x are complex numbers.

The Needles

The needles are those $(2F1)$'s that are evaluable in terms of the Gamma function, or more precisely, those for which the sequence

$$u_n := F(-an, bn + b_0, cn + c_0, x) \quad ,$$

is a *hypergeometric* sequence which means that

$$r_n := \frac{u_{n+1}}{u_n} \quad ,$$

is a *rational function* of n, i.e.

$$r_n = \frac{P(n)}{Q(n)} \quad ,$$

where $P(n)$ and $Q(n)$ are *polynomials* in n. It is easy to see, by looking at the asymptotics, that the degrees of P and Q must be the same.

How to Test Whether Something is a Needle or just a Boring piece of Hay?

Even with a *specific* choice of a, b, c, b_0, c_0, x, there is no way, *a priori*, to rule out (conclusively) the possibility that the sequence u_n is hypergeometric, since, who knows?, the degree d could be a zillion. But if we restrict the search for some fixed (not too big!) d, then it is very easy (with computers, of course), to decide whether the studied sequence u_n is hypergeometric with the degree of both top and bottom of the r_n being $\leq d$.

Indeed, write $r_n = P(n)/Q(n)$ *generically*,

$$r_n = \frac{\sum_{i=0}^{d} p_i n^i}{\sum_{i=0}^{d} q_i n^i} \quad ,$$

in terms of the $2d + 2$ **undetermined coefficients** $p_0, p_1, \dots p_d$, $q_0, q_1, \dots, q_d$, and plug-in $n = 0, 1, 2, \dots, 2d + 6$, say. You would get $2d + 7$ equations for the $2d + 2$ unknowns, and short of a miracle, they would not be solvable. If there are solvable, then with probability $1 - 10^{-10000}$, r_n is indeed the conjectured rational function, and if you want to have it true with probability 1, then all you need to do is find the corresponding u_n (that solves $u_{n+1}/u_n = r_n$), and then use Zeilberger's algorithm to prove the conjectured identity rigorously.

[The above is done in procedure `NakhD` in our Maple package `BruteTwoFone`. For example, to guess the Chu-Vandermonde identity, type `NakhD([[-n,a],[c],1],n,1);`, and you would get
$(c + n - a)/(c + n)$]

Alas the Haystack is infinite

Even with a specific a and a specific degree d, there are *infinitely* many things to try. However, if we leave b, b_0, c, c_0, x *symbolic*, then for any specific, numeric n_0, u_{n_0} will no longer be a mere number, but a certain *rational function* of (b, b_0, c, c_0, x).

We are looking for those choices of the parameters (b, b_0, c, c_0, x), for which the linear equations in the coefficients of r_n, namely $p_0, \dots, p_d$, $q_0, \dots, q_d$, are solvable. This means, using linear algebra, that certain determinants vanish. We can add as many conditions as we want, by finding the determinants corresponding to the set of equation $P(i)/Q(i) = r_i$, for $i = 1, \dots, M$, for M big enough to have more equations than unknowns. Then using the Buchberger algorithm, we can solve them, and get *all* the choices of b, b_0, c, c_0, x (including infinite families, for example, we should get $x = 1$ to account for the Chu-Vandermonde identity).

Alas our Computers are not Big Enough

Unfortunately, the above scheme is not feasible, since the equations are *sooo* huge, and Buchberger's algorithm is *sooo* slow. So we have to compromise. We *fix* b, b_0, c, c_0, and search for the lucky x.

Now we only need two determinants, both being certain polynomials in the single variable x, and simply take their greatest common divisor (`gcd` in Maple), to get those x that (have the potential, and probably will) yield closed-form evaluations (with the given d) for those *fixed* a, b, c, b_0, c_0.

[The above is done in procedure `NakhDx` in our Maple package `BruteTwoFone`. For example, to guess Theorem 12 of [E], type

`NakhDx([-n,-3*n-1],[-2*n],n,10);`,

and you would get $x = (1 \pm \sqrt{3}i)/2$, for the two choices for x that would make $F(-n, -3n-1, -2n, x)$ hypergeometric of degree ≤ 10.]

The Big Five-Fold Do-Loop

So we decide beforehand on an M, and a denominator D, and try *all* $F(-an, bn+b_0, cn+c_0, x)$ (x yet-to-be-determined) with integers a, b, c such that $1 \leq a \leq M$, and $-M \leq b, c \leq M$, and rational numbers b_0, c_0 with denominator D such that $-M \leq b_0, c_0 \leq M$ and wait for the luck-of-the-draw.

Removing the Chaff

Of course $x = 1$ is just Gauss. There are also Kummers with $x = -1$ and their associates via the transformation rules that have to be removed.

The Wheat

The output with $M = 4$ and $D \leq 4$ and $d = 6$ was rather numerous, but most of them turned out to be special cases of the following two *infinite* families of strange evaluations.

Theorem 1. For any non-negative integer r, we have

$$F(-2n, b, -2n+2r-b, -1) = \frac{(\frac{1}{2})_n (b+1-r)_n}{(b/2+1-r)_n (b/2+\frac{1}{2}-r)_n} \cdot \left(\sum_{i=0}^{r-1} \frac{2^{2i} i! \binom{r+i-1}{2i}}{(b-r+1)_i} \cdot \binom{n}{i} \right) \quad . \tag{$PerturbedKummer$}$$

Note. There is an analogous statement with $2r$ replaced by $2r+1$ that we omit. There are also analogous statements for $F(-2n, b+r, -2n-b, -1)$ that we also omit).

Three Sketches of Three Proofs

First Proof. (Shalosh B. Ekhad) For each *specific* r this is immediately doable by the Zeilberger algorithm, but even the general case is doable that way by dividing both sides of ($PerturbedKummer$) by

$$\frac{(\frac{1}{2})_n (b+1-r)_n}{(b/2+1-r)_n (b/2+\frac{1}{2}-r)_n} \quad ,$$

leaving a polynomial on the right side. Now apply the Zeilberger algorithm to this new left side, getting a certain second-order linear recurrence operator that annihilates this new left side (leaving

r symbolic), and then verifying that the polynomial on the right side indeeed annihilates it (and checking the trivial initial conditions).

Second Proof. (Dennis Stanton) Use an appropriate specialization of Eq. (1), Sec. 4.7 of [B], and iterate.

Third Proof. (Christian Krattenthaler) Using his versatile package HYP ([K]), Krattenthaler used contiguous relations. See his write-up

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/ck.pdf`,

that he kindly allowed us to post.

Conjecture 1. For any integers i and j,

$$F(-2n, -\frac{1}{2} + i, -3n - \frac{1}{2} + j, -3)$$

is evaluable in closed-form.

Comment. We can easily find the explicit forms for every specific i and j. The list of all those exact evaluations for $-5 \leq i, j \leq 5$ can be found in:

`http://www.math.rutgers.edu/~zeilberg/tokhniot/findhg/oApaZclosedForm` .

However, we were unable to find a *uniform* expression in terms of i and j, like in Theorem 1. It is very possible that it can be proved (perhaps without being able to write it uniformly), by using contiguous relations, as done in Krattenthaler's proof above.

The Remaining Strange Identities

After removing all the identities covered by Theorem 1 and Conjecture 1, 19 (inequivalent) strange hypergeometric identities remained. Most of them are already in [E], so we won't list them here, but refer the reader to `PreComputed21();` in our package `BruteTwoFone`, and, in a more human-readable form, complete with the evaluations to:

`http://www.math.rutgers.edu/~zeilberg/tokhniot/findhg/oSefer21` .

Acknowledgement

This paper was inspired by an intriguing question posed by John Greene (to find a one-parameter family generalizing the exact evaluation, in terms of the Gamma function evaluated at rational arguments, of $F(\frac{1}{4}, \frac{3}{4}, 1, -\frac{1}{63})$) that unfortunately we were unable to answer. We also thank Dennis Stanton and Christian Krattenthaler for supplying their proofs of Theorem 1, and last but not least, we would like to thank Shalosh B. Ekhad for stimulating discussions and insight.

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First Written: Feb. 22, 2008

This version: Feb. 22, 2008